Favard length and quantitative rectifiability

Damian Dąbrowski

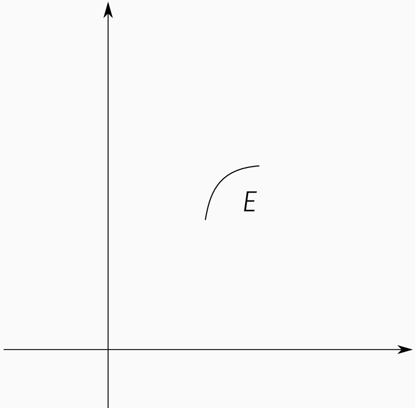
JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Research Council of Finland

Vitushkin’s conjecture

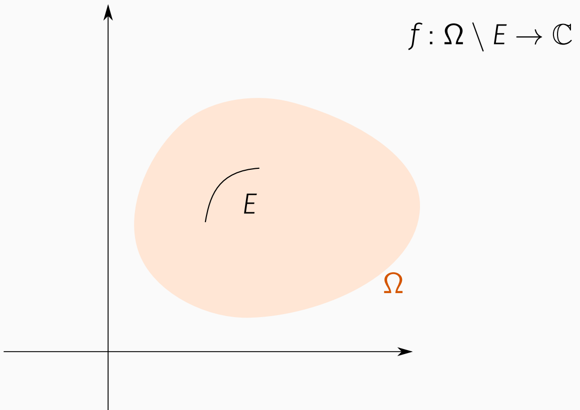
Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



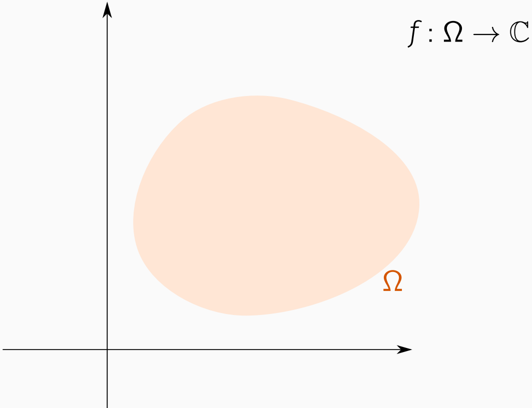
Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



Analytic capacity

In 1947 Ahlfors characterized removability in terms of analytic capacity:

$$E \text{ is removable} \quad \Leftrightarrow \quad \gamma(E) = 0,$$

where

$$\gamma(E) = \sup\{|f'(\infty)| \; : \; f : \mathbb{C} \setminus E \to \mathbb{C} \text{ analytic}, \; \|f\|_\infty \leq 1\},$$

$$f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty)).$$

Painlevé problem

Painlevé problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Painlevé problem

Painlevé problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Classical:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.

Painlevé problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Classical:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.
- If E is a segment, then $\gamma(E) = c\,\mathcal{H}^1(E)$.

Painlevé problem

Painlevé problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Classical:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.
- If E is a segment, then $\gamma(E) = c\,\mathcal{H}^1(E)$.

Question

$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$?

Painlevé problem

Painlevé problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Classical:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.
- If E is a segment, then $\gamma(E) = c\,\mathcal{H}^1(E)$.

Question

$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$? No!

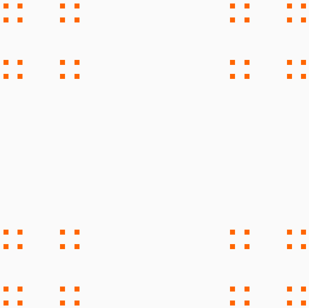
There are sets $E \subset \mathbb{C}$ with $\gamma(E) = 0$ and $0 < \mathcal{H}^1(E) < \infty$. (Vitushkin 1959, Garnett, Ivanov 1970s)

Vitushkin's conjecture

The sets constructed by Vitushkin, Garnett and Ivanov had very small projections. More precisely, they satisfied

$$\mathcal{H}^1(\pi_\theta(E)) = 0$$

for a.e. direction $\theta \in [0, \pi]$.



Vitushkin's conjecture

The sets constructed by Vitushkin, Garnett and Ivanov had very small projections. More precisely, they satisfied

$$\mathcal{H}^1(\pi_\theta(E)) = 0$$

for a.e. direction $\theta \in [0, \pi]$.

Define **Favard length** of E as

$$\text{Fav}(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

Vitushkin's conjecture (1967)

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \text{Fav}(E) = 0$$

Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

- In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)

Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

- In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)
- In the case $\mathcal{H}^1(E) = \infty$, Vitushkin's conjecture is **false** (Mattila '86, Jones-Murai '88):

$$\mathrm{Fav}(E) = 0 \quad \not\Rightarrow \quad \gamma(E) = 0.$$

Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

- In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)
- In the case $\mathcal{H}^1(E) = \infty$, Vitushkin's conjecture is false (Mattila '86, Jones-Murai '88):
$$\mathrm{Fav}(E) = 0 \quad \not\Rightarrow \quad \gamma(E) = 0.$$
- What about
$$\mathrm{Fav}(E) = 0 \quad \Leftarrow \quad \gamma(E) = 0?$$

Open problems

Problem 1 (qualitative)

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Open for sets $E \subset \mathbb{C}$ with $\dim_H(E) = 1$ and non-σ-finite $\mathcal{H}^1$-measure.

Open problems

Problem 1 (qualitative)

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Open for sets $E \subset \mathbb{C}$ with $\dim_H(E) = 1$ and non-σ-finite $\mathcal{H}^1$-measure.

Problem 2 (quantitative)

$$\gamma(E) \gtrsim \mathrm{Fav}(E)?$$

$$\gamma(E) \gtrsim_{\mathrm{Fav}(E)} 1?$$

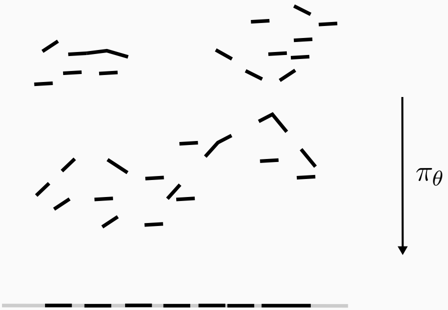
Open even for sets with finite length.

What happens for sets with finite length?

Geometric ingredient:

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$,

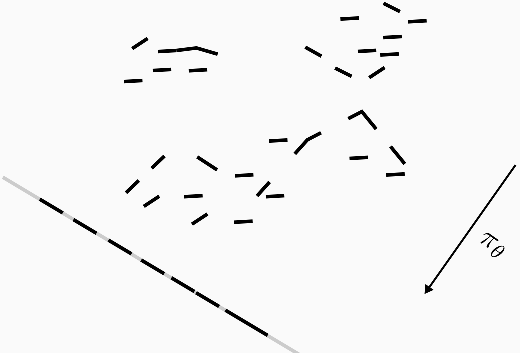


Two ingredients

Geometric ingredient:

Theorem (Besicovitch 1939)

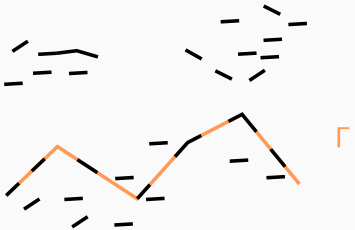
Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$,



Geometric ingredient:

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(E \cap \Gamma) > 0$.



Two ingredients

Geometric ingredient:

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\text{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathcal{H}^1(E \cap \Gamma) > 0$.

Analytic ingredient:

Theorem (Calderón 1977)

If Γ is a rectifiable curve and $F \subset \Gamma$ satisfies $\mathcal{H}^1(F) > 0$, then

$$\gamma(F) > 0.$$

Vitushkin's conjecture when $\mathcal{H}^1(E) < \infty$

Goal

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0$$

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$

$$\gamma(E) \geq \gamma(E \cap \Gamma)$$

Vitushkin's conjecture when $\mathcal{H}^1(E) < \infty$

Goal

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0$$

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0.$$

■

Vitushkin's conjecture when $\mathcal{H}^1(E) < \infty$

Goal

$$\text{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0$$

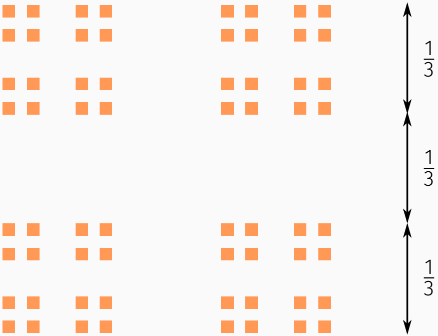
If $0 < \mathcal{H}^1(E) < \infty$ and $\text{Fav}(E) > 0$, then by the Besicovitch projection theorem $\exists \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0. \qquad \blacksquare$$

- Why does it only work for sets with finite length?
- Why does it give no quantitative estimates?

First problem

The Besicovitch projection theorem **fails** for sets with infinite length!



$K = C_{1/3} \times C_{1/3}$ satisfies $\mathrm{Fav}(K) \gtrsim 1$ and $\mathcal{H}^1(K \cap \Gamma) = 0$ for every rectifiable curve.

Second problem

Recall: if $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$ and

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0.$$

Second problem

Recall: if $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$ and

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0.$$

There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

$$\gamma(E \cap \Gamma) \gtrsim_L \mathcal{H}^1(E \cap \Gamma)...$$

Second problem

Recall: if $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then $\exists\, \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$ and

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0.$$

There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

$$\gamma(E \cap \Gamma) \gtrsim_L \mathcal{H}^1(E \cap \Gamma)...$$

...but the Besicovitch projection theorem gives **no quantitative bound** neither on $\mathcal{H}^1(E \cap \Gamma)$, nor on $\mathrm{Lip}(\Gamma)$!

Second problem

Recall: if $0 < \mathcal{H}^1(E) < \infty$ and $\mathbf{Fav}(E) > 0$, then $\exists \Gamma$ with $\mathcal{H}^1(E \cap \Gamma) > 0$ and

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón)}}{>} 0.$$

There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

$$\gamma(E \cap \Gamma) \gtrsim_L \mathcal{H}^1(E \cap \Gamma)...$$

...but the Besicovitch projection theorem gives **no quantitative bound** neither on $\mathcal{H}^1(E \cap \Gamma)$, nor on $\mathrm{Lip}(\Gamma)$!

Favard length problem

Can we quantify the dependence of $\mathbf{Lip}(\Gamma)$ and $\mathcal{H}^1(E \cap \Gamma)$ on $\mathbf{Fav}(E)$?

Favard length problem

Naive conjecture...

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathbf{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

Naive conjecture

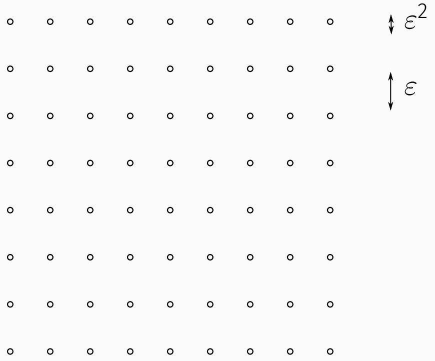
Let $E \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathbf{Fav}(E) \gtrsim 1$. Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

… is false

For any $\varepsilon > 0$ there exists a set $E = E_\varepsilon \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathsf{Fav}(E) \gtrsim 1$ such that for all L-Lipschitz graphs Γ

$$\mathcal{H}^1(E \cap \Gamma) \lesssim L\varepsilon.$$

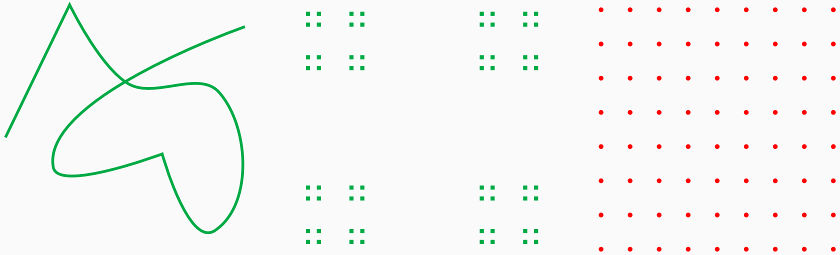


E consists of ε^{-2} uniformly distributed circles of radius ε^2.

Reasonable conjecture

We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for every $x \in E$ and $0 < r < \operatorname{diam}(E)$

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x,r)) \leq Cr.$$



Reasonable conjecture

We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for every $x \in E$ and $0 < r < \operatorname{diam}(E)$

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x,r)) \leq Cr.$$

Reasonable conjecture

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

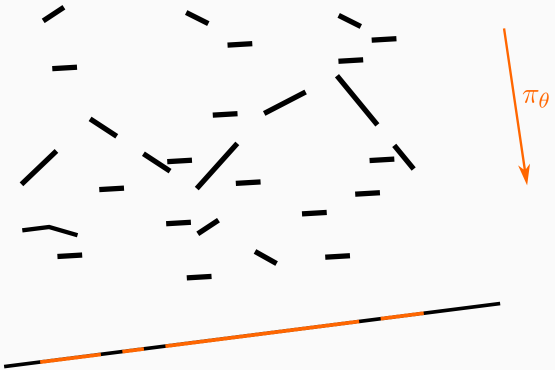
Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Variations on this conjecture appearing since the 90s in the works of David and Semmes, Mattila, Peres and Solomyak.

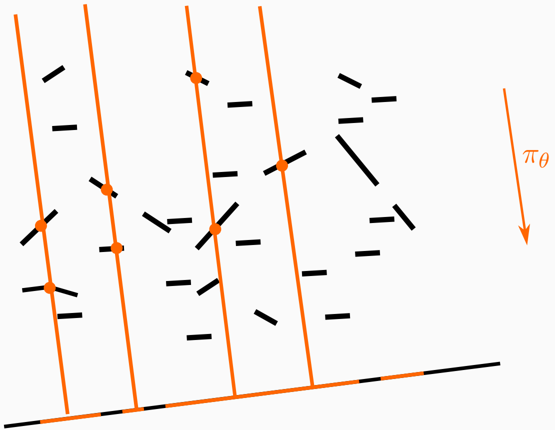
What is this really about?

big projections



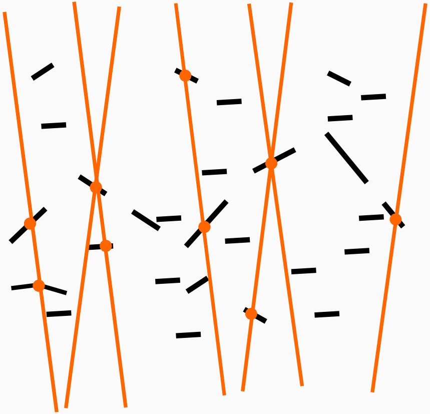
What is this really about?

big projections $\Rightarrow$ many lines with few intersections



What is this really about?

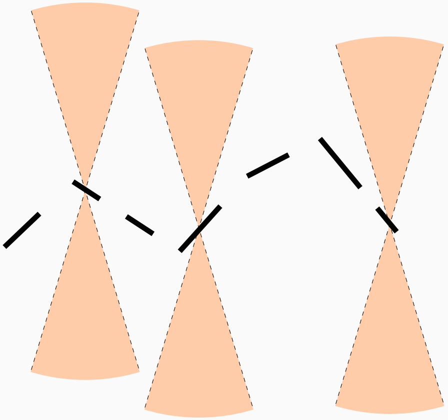
big projections $\Rightarrow$ many lines with few intersections



What is this really about?

big projections $\Rightarrow$ many lines with few intersections

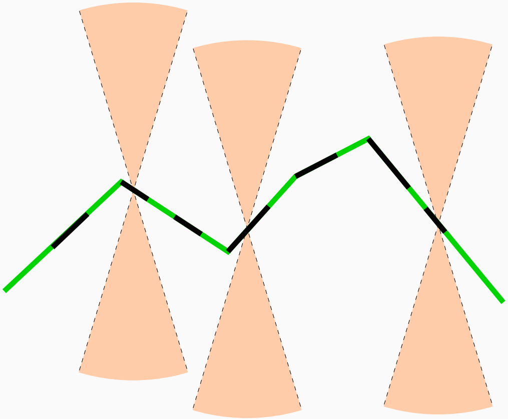
$\Rightarrow$ cones with no intersections



What is this really about?

big projections $\Rightarrow$ many lines with few intersections

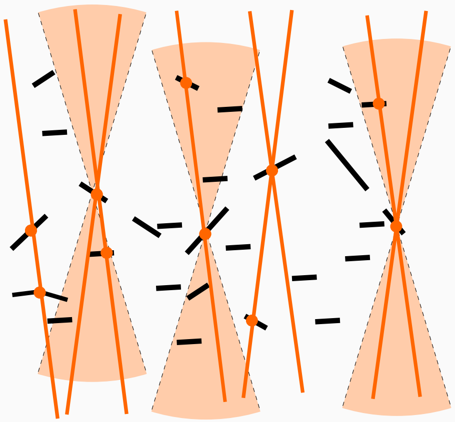
$\Rightarrow$ cones with no intersections $\Rightarrow$ subset of a Lipschitz graph



What is this really about?

big projections ⇒ many lines with few intersections

⇒ cones with no intersections ⇒ subset of a Lipschitz graph



Reasonable conjecture

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathbf{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Progress on the conjecture consisted of replacing "$\mathbf{Fav}(E) \gtrsim \mathcal{H}^1(E)$" by:

- **David-Semmes '93**: big projection + WGL
- **Martikainen-Orponen '18**: projections in L^2
- **Orponen '21**: plenty of big projections
- **D. '22**: projections in L^∞

New result: the conjecture is true!

Theorem (D. '24)

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

New result: the conjecture is true!

Theorem (D. '24)

Let $E \subset \mathbb{R}^2$ be an Ahlfors regular set with $\mathbf{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Corollaries:

- a positive answer to a 1993 question of David and Semmes,
- a positive answer to a 2002 question of Peres and Solomyak,
- progress on Vitushkin's conjecture.

Back to Vitushkin

Estimates for Ahlfors regular sets

Quantitative Vitushkin's conjecture

If $E \subset \mathbb{R}^2$ is compact and $\mathrm{Fav}(E) \geq \kappa \, \mathrm{diam}(E)$, do we have

$$\gamma(E) \gtrsim_\kappa \mathrm{diam}(E)?$$

Partial results in Chang-Tolsa '20, Tasso '22, D.-Villa '22.

Estimates for Ahlfors regular sets

Quantitative Vitushkin's conjecture

If $E \subset \mathbb{R}^2$ is compact and $\mathrm{Fav}(E) \geq \kappa\,\mathrm{diam}(E)$, do we have

$$\gamma(E) \gtrsim_\kappa \mathrm{diam}(E)?$$

Partial results in Chang-Tolsa '20, Tasso '22, D.-Villa '22.

Corollary (D. '24)

If $E \subset \mathbb{R}^2$ is Ahlfors regular and $\mathrm{Fav}(E) \geq \kappa\,\mathrm{diam}(E)$, then

$$\gamma(E) \gtrsim_\kappa \mathrm{diam}(E).$$

Sets with uniformly large Favard length

We say that a set $E \subset \mathbb{R}^2$ has **uniformly large Favard length** if it is compact and for all $x \in E$ and $0 < r < \text{diam}(E)$

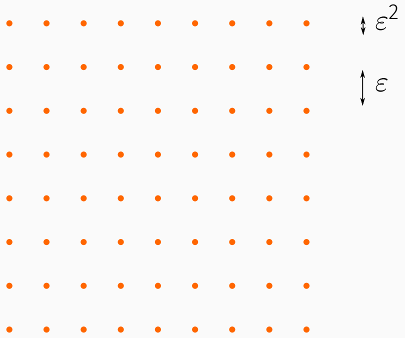
$$\text{Fav}(E \cap B(x, r)) \geq \kappa r.$$

Sets with ULFL

Sets with uniformly large Favard length

We say that a set $E \subset \mathbb{R}^2$ has **uniformly large Favard length** if it is compact and for all $x \in E$ and $0 < r < \mathrm{diam}(E)$

$$\mathrm{Fav}(E \cap B(x, r)) \geq \kappa r.$$

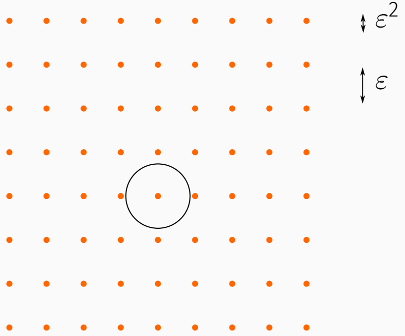


A set violating ULFL

Sets with uniformly large Favard length

We say that a set $E \subset \mathbb{R}^2$ has uniformly large Favard length if it is compact and for all $x \in E$ and $0 < r < \operatorname{diam}(E)$

$$\operatorname{Fav}(E \cap B(x, r)) \geq \kappa r.$$



A set violating ULFL

Sets with uniformly large Favard length

We say that a set $E \subset \mathbb{R}^2$ has **uniformly large Favard length** if it is compact and for all $x \in E$ and $0 < r < \mathrm{diam}(E)$

$$\mathrm{Fav}(E \cap B(x, r)) \geq \kappa r.$$

Corollary (D. '24 + D.-Villa '22)

If $E \subset \mathbb{R}^2$ has ULFL, then

$$\gamma(E) \gtrsim_\kappa \mathrm{diam}(E).$$

Proof of the main result

Goal

Theorem (D. '24)

Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathsf{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathsf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Goal

Theorem (D. '24)

Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathbf{Fav}(E) \gtrsim \mathcal{H}^1(E)$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

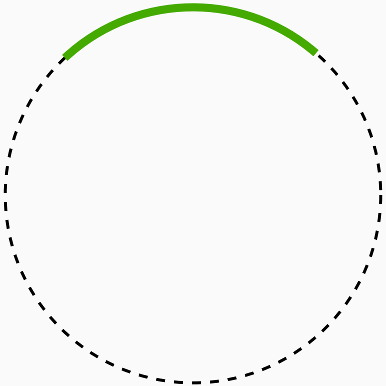
$$\mathcal{H}^1(E \cap \Gamma) \gtrsim \mathcal{H}^1(E).$$

Key tool: conical energies introduced in [Martikainen-Orponen '18] and [Chang-Tolsa '20].

For any $\theta \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set $\ell_{x,\theta} := x + \mathrm{span}(\theta)$.

Given $G \subset \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set

$$X(x, G) := \bigcup_{\theta \in G} \ell_{x,\theta}.$$

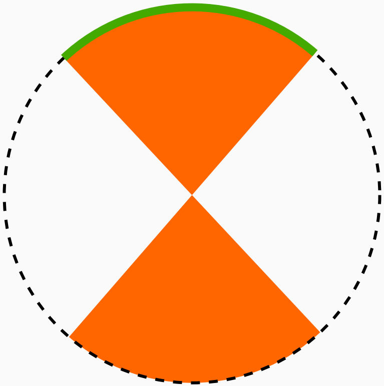


Cones

For any $\theta \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set $\ell_{x,\theta} := x + \text{span}(\theta)$.

Given $G \subset \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set

$$X(x, G) := \bigcup_{\theta \in G} \ell_{x,\theta}.$$

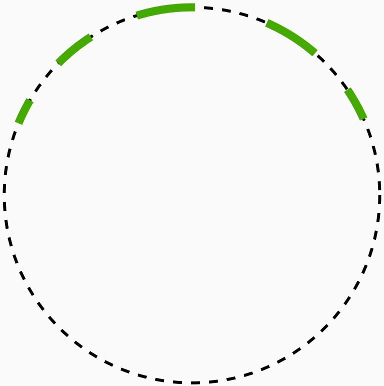


Cones

For any $\theta \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set $\ell_{x,\theta} := x + \mathrm{span}(\theta)$.

Given $G \subset \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set

$$X(x, G) := \bigcup_{\theta \in G} \ell_{x,\theta}.$$

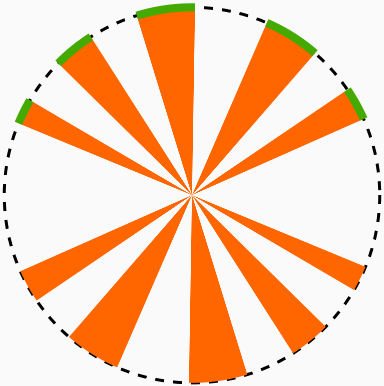


Cones

For any $\theta \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set $\ell_{x,\theta} := x + \mathrm{span}(\theta)$.

Given $G \subset \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set

$$X(x, G) := \bigcup_{\theta \in G} \ell_{x,\theta}.$$



Cones

For any $\theta \in \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set $\ell_{x,\theta} := x + \mathrm{span}(\theta)$.

Given $G \subset \mathbb{S}^1$ and $x \in \mathbb{R}^2$ set

$$X(x, G) := \bigcup_{\theta \in G} \ell_{x,\theta}.$$

Given $0 < r < R$ we define the truncated cones

$$X(x, G, r) := X(x, G) \cap B(x, r)$$

and

$$X(x, G, r, R) := X(x, G, R) \setminus B(x, r).$$

Conical energies

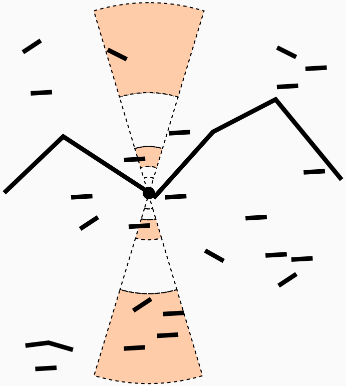
Given $x \in \mathbb{R}^2$, $G \subset \mathbb{S}^1$, and a measure μ we define the **conical energy of μ at x** as

$$\mathcal{E}_\mu(x, G) = \int_0^\infty \frac{\mu(X(x, G, r))}{r} \frac{dr}{r}$$

Conical energies

Given $x \in \mathbb{R}^2$, $G \subset \mathbb{S}^1$, and a measure μ we define the **conical energy of μ at x** as

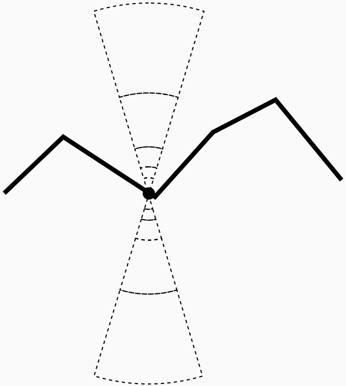
$$\mathcal{E}_\mu(x, G) = \int_0^\infty \frac{\mu(X(x, G, r))}{r} \frac{dr}{r} \sim \sum_{k \in \mathbb{Z}} \frac{\mu(X(x, G, 2^{-k}, 2^{-k+1}))}{2^{-k}}.$$



Note: if $\mathcal{E}_\mu(x, J) = 0$ for μ-a.e. x with a fixed arc $J \subset \mathbb{S}^1$, then

$$\mu(X(x, J)) = 0 \quad \text{for } \mu\text{-a.e. } x,$$

and so μ is concentrated on a Lipschitz graph.



Finding Lipschitz graphs

Note: if $\mathcal{E}_\mu(x, J) = 0$ for μ-a.e. x with a fixed arc $J \subset \mathbb{S}^1$, then

$$\mu(X(x, J)) = 0 \quad \text{for } \mu\text{-a.e. } x,$$

and so μ is concentrated on a Lipschitz graph.

Theorem (Martikainen-Orponen '18)

Assume that $E \subset B(0,1)$ is Ahlfors regular, $F \subset E$ with $\mathcal{H}^1(F) \sim \mathcal{H}^1(E)$, and there exists an arc $J \subset \mathbb{S}^1$ with $\mathcal{H}^1(J) \gtrsim 1$ such that for $\mu = \mathcal{H}^1|_F$

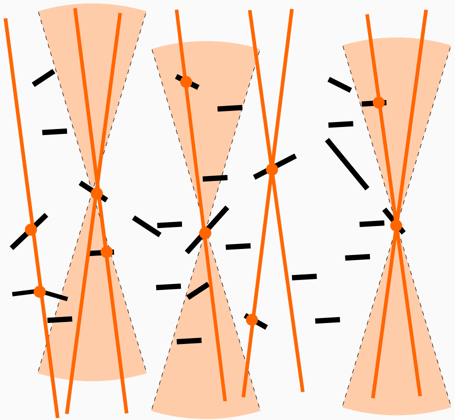
$$\mathcal{E}_\mu(x, J) \lesssim 1.$$

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and $\mathcal{H}^1(F \cap \Gamma) \gtrsim 1$.

From big projections to conical energies

big projections ⇒ many lines with few intersections

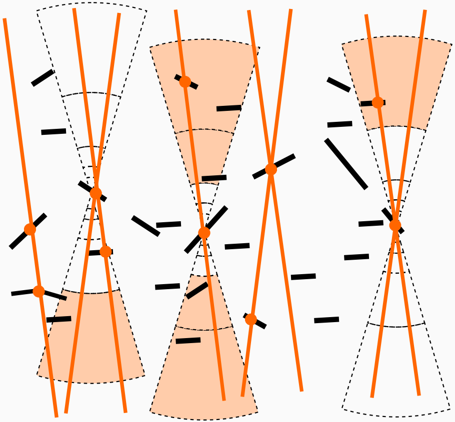
⇒ cones with no intersections ⇒ subset of a Lipschitz graph



From big projections to conical energies

big projections $\Rightarrow$ many lines with few intersections

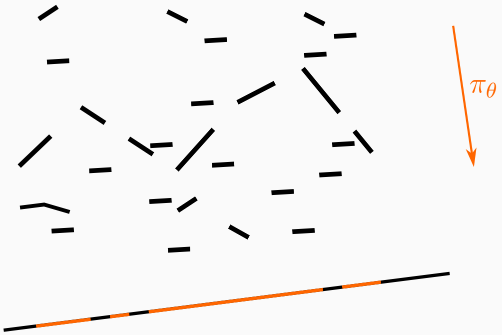
$\Rightarrow$ bounded conical energies $\overset{\text{[MO18]}}{\Rightarrow}$ subset of a Lipschitz graph



“Many lines with few intersections”

Lemma

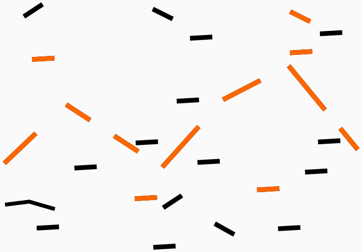
Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathsf{Fav}(E) \gtrsim \mathcal{H}^1(E)$.



“Many lines with few intersections”

Lemma

Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathrm{Fav}(E) \gtrsim \mathcal{H}^1(E)$. Then, there exists $F \subset E$ with $\mathcal{H}^1(F) \sim \mathcal{H}^1(E)$

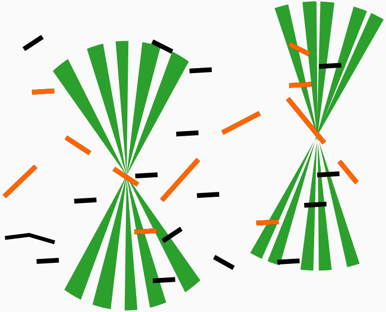


"Many lines with few intersections"

Lemma

Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathbf{Fav}(E) \gtrsim \mathcal{H}^1(E)$. Then, there exists $F \subset E$ with $\mathcal{H}^1(F) \sim \mathcal{H}^1(E)$ such that for every $x \in F$ there is $G(x) \subset \mathbb{S}^1$ with $\mathcal{H}^1(G(x)) \gtrsim 1$ and

$$\mathcal{E}_\mu(x, G(x)) \lesssim 1.$$



“Many lines with few intersections”

Lemma

Let $E \subset B(0,1)$ be an Ahlfors regular set with $\mathbf{Fav}(E) \gtrsim \mathcal{H}^1(E)$. Then, there exists $F \subset E$ with $\mathcal{H}^1(F) \sim \mathcal{H}^1(E)$ such that for every $x \in F$ there is $G(x) \subset \mathbb{S}^1$ with $\mathcal{H}^1(G(x)) \gtrsim 1$ and

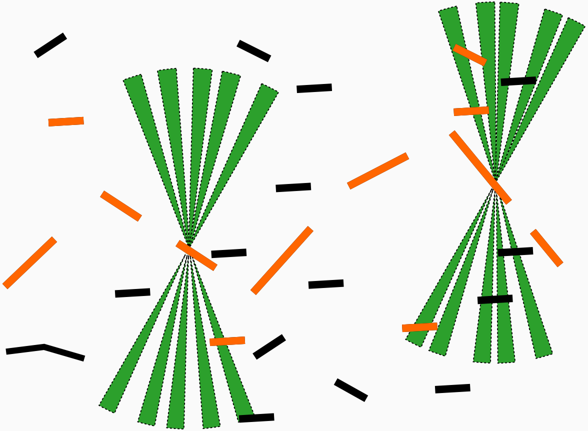
$$\mathcal{E}_\mu(x, G(x)) \lesssim 1.$$

This is close to [MO18], but there are two problems:

- $G(x) \subset \mathbb{S}^1$ might not be an arc,
- $G(x)$ depends on the point x.

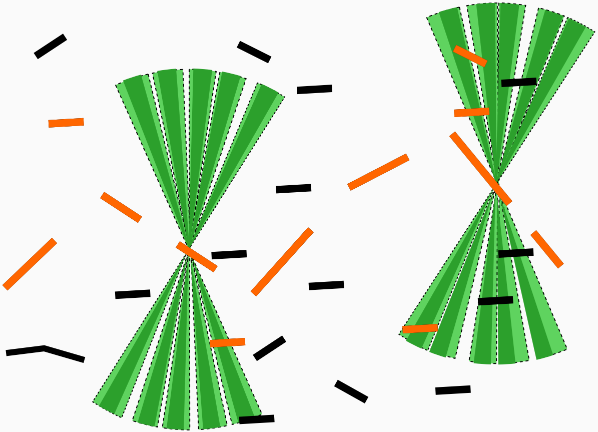
[MO18] requires that $G(x) = J$ for some fixed arc $J \subset \mathbb{S}^1$.

$$\int_F \mathcal{E}_\mu(x, G(x))\, d\mu(x) \lesssim \mu(F).$$



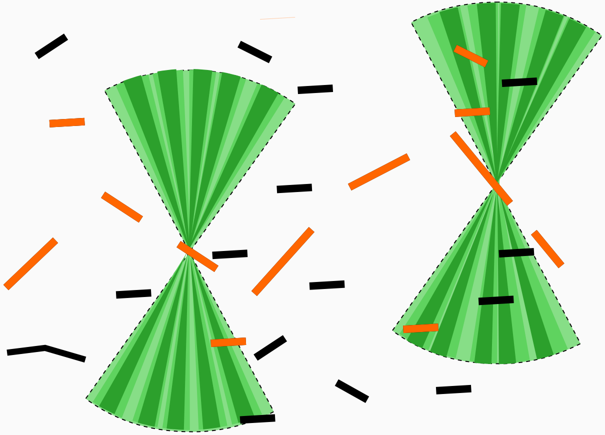
Good directions propagate

$$\int_F \mathcal{E}_\mu(x, G_*(x))\, d\mu(x) \lesssim \int_F \mathcal{E}_\mu(x, G(x))\, d\mu(x) \lesssim \mu(F).$$



Good directions propagate

$$\int_F \mathcal{E}_\mu(x, J)\, d\mu(x) \lesssim \int_F \mathcal{E}_\mu(x, G_*(x))\, d\mu(x) \lesssim \int_F \mathcal{E}_\mu(x, G(x))\, d\mu(x) \lesssim \mu(F).$$



Good directions propagate

$$\int_F \mathcal{E}_\mu(x, J)\, d\mu(x) \lesssim \int_F \mathcal{E}_\mu(x, G_*(x))\, d\mu(x) \lesssim \int_F \mathcal{E}_\mu(x, G(x))\, d\mu(x) \lesssim \mu(F).$$

Proof of the main result:

Lemma + Propagation + [MO18] = big piece of a Lipschitz graph.

Toy version of the propagation result

Proposition

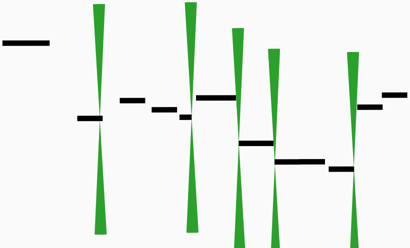
Let $E \subset B(0,1)$ be an Ahlfors regular set consisting of parallel segments.

Toy version of the propagation result

Proposition

Let $E \subset B(0,1)$ be an Ahlfors regular set consisting of parallel segments. Assume that there is an arc $J \subset \mathbb{S}^1$ "parallel" to the segments such that

$$E \cap X(x, J) = \{x\} \quad \text{for } x \in E.$$



Toy version of the propagation result

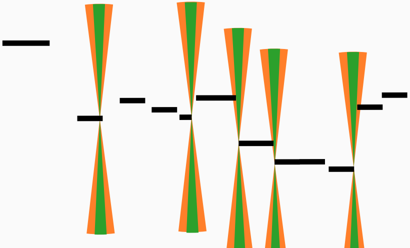
Proposition

Let $E \subset B(0,1)$ be an Ahlfors regular set consisting of parallel segments. Assume that there is an arc $J \subset \mathbb{S}^1$ "parallel" to the segments such that

$$E \cap X(x, J) = \{x\} \quad \text{for } x \in E.$$

Then,

$$\int \mathcal{E}_\mu(x, 3J)\, d\mu(x) \lesssim \mathcal{H}^1(J)\mu(E).$$



Toy version of the propagation result

Proposition

Let $E \subset B(0,1)$ be an Ahlfors regular set consisting of parallel segments. Assume that there is an arc $J \subset \mathbb{S}^1$ "parallel" to the segments such that

$$E \cap X(x, J) = \{x\} \quad \text{for } x \in E.$$

Then,

$$\int \mathcal{E}_\mu(x, 3J)\, d\mu(x) \lesssim \mathcal{H}^1(J)\mu(E).$$

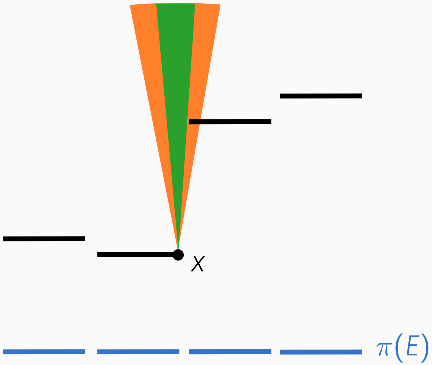
$$\mathcal{E}_\mu(x, 3J) = \mathcal{E}_\mu(x, 3J \setminus J) \sim \sum_{k \in \mathbb{Z}} \frac{\mu(X(x, 3J \setminus J, 2^{-k}, 2^{-k+1}))}{2^{-k}}$$

$$= \sum_{k \in \mathrm{Bad}(x)} \frac{\mu(X(x, 3J \setminus J, 2^{-k}, 2^{-k+1}))}{2^{-k}} \sim \mathcal{H}^1(J) \cdot \#\mathrm{Bad}(x).$$

Key geometric lemma

If $k \in \mathrm{Bad}(x)$, then there exists a "gap" I in $\pi(E)$ such that

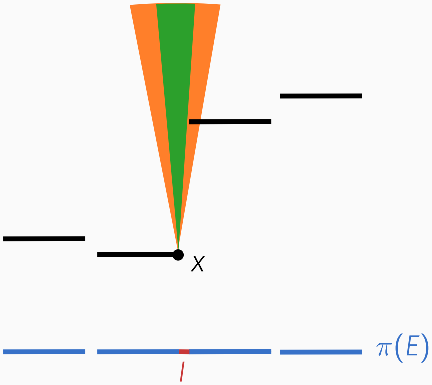
$$\mathcal{H}^1(I) \sim \mathcal{H}^1(J) \cdot 2^{-k} \quad \text{and} \quad \pi(x) \in 5I.$$



Key geometric lemma

If $k \in \text{Bad}(x)$, then there exists a "gap" I in $\pi(E)$ such that

$$\mathcal{H}^1(I) \sim \mathcal{H}^1(J) \cdot 2^{-k} \quad \text{and} \quad \pi(x) \in 5I.$$



Key geometric lemma

If $k \in \mathrm{Bad}(x)$, then there exists a "gap" I in $\pi(E)$ such that

$$\mathcal{H}^1(I) \sim \mathcal{H}^1(J) \cdot 2^{-k} \quad \text{and} \quad \pi(x) \in 5I.$$

The lemma implies the proposition:

$$\int_E \mathcal{E}_\mu(x, 3J)\, d\mu(x) \sim \mathcal{H}^1(J) \int_E \#\mathrm{Bad}(x)\, d\mu(x)$$

Key geometric lemma

If $k \in \mathrm{Bad}(x)$, then there exists a "gap" I in $\pi(E)$ such that

$$\mathcal{H}^1(I) \sim \mathcal{H}^1(J) \cdot 2^{-k} \quad \text{and} \quad \pi(x) \in 5I.$$

The lemma implies the proposition:

$$\begin{aligned}\int_E \mathcal{E}_\mu(x, 3J)\, d\mu(x) &\sim \mathcal{H}^1(J) \int_E \#\mathrm{Bad}(x)\, d\mu(x) \\ &= \mathcal{H}^1(J) \sum_{k \geq 0} \mu(\{x \in E : k \in \mathrm{Bad}(x)\})\end{aligned}$$

Key geometric lemma

If $k \in \mathrm{Bad}(x)$, then there exists a "gap" I in $\pi(E)$ such that

$$\mathcal{H}^1(I) \sim \mathcal{H}^1(J) \cdot 2^{-k} \quad \text{and} \quad \pi(x) \in 5I.$$

The lemma implies the proposition:

$$\begin{aligned} \int_E \mathcal{E}_\mu(x, 3J)\, d\mu(x) &\sim \mathcal{H}^1(J) \int_E \#\mathrm{Bad}(x)\, d\mu(x) \\ &= \mathcal{H}^1(J) \sum_{k \geq 0} \mu(\{x \in E : k \in \mathrm{Bad}(x)\}) \\ &\overset{\text{KGL}}{\lesssim} \mathcal{H}^1(J) \sum_{k \geq 0} \sum_{\substack{I \in \mathrm{Gap}, \\ \mathcal{H}^1(I) \sim \mathcal{H}^1(J) 2^{-k}}} \mu(\pi^{-1}(5I)) \end{aligned}$$

Key geometric lemma

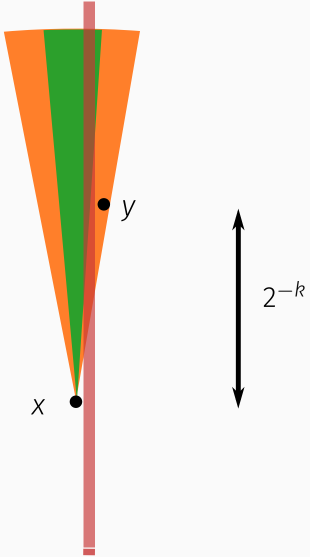
If $k \in \mathrm{Bad}(x)$, then there exists a "gap" I in $\pi(E)$ such that

$$\mathcal{H}^1(I) \sim \mathcal{H}^1(J) \cdot 2^{-k} \quad \text{and} \quad \pi(x) \in 5I.$$

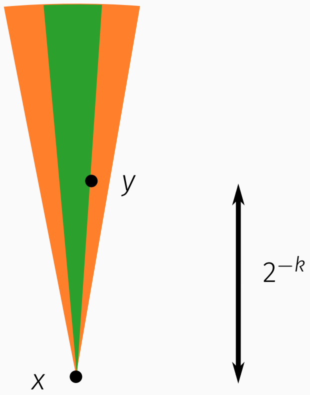
The lemma implies the proposition:

$$\begin{aligned}
\int_E \mathcal{E}_\mu(x, 3J)\, d\mu(x) \sim \mathcal{H}^1(J) \int_E \#\mathrm{Bad}(x)\, d\mu(x) \\
&= \mathcal{H}^1(J) \sum_{k \geq 0} \mu(\{x \in E : k \in \mathrm{Bad}(x)\}) \\
&\overset{\text{KGL}}{\lesssim} \mathcal{H}^1(J) \sum_{k \geq 0} \sum_{\substack{I \in \mathrm{Gap}, \\ \mathcal{H}^1(I) \sim \mathcal{H}^1(J) 2^{-k}}} \mu(\pi^{-1}(5I)) \\
&\sim \mathcal{H}^1(J) \sum_{k \geq 0} \sum_{\substack{I \in \mathrm{Gap}, \\ \mathcal{H}^1(I) \sim \mathcal{H}^1(J) 2^{-k}}} \mathcal{H}^1(I) \lesssim \mathcal{H}^1(J)\, \mathrm{diam}(E).
\end{aligned}$$

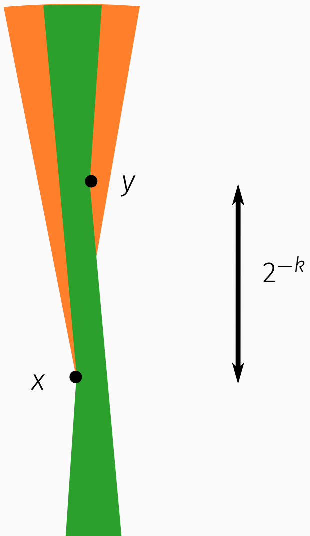
“Proof” of the key geometric lemma



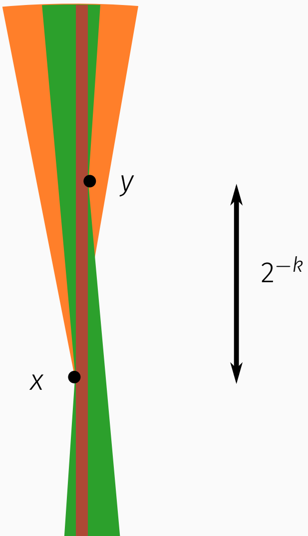
“Proof” of the key geometric lemma



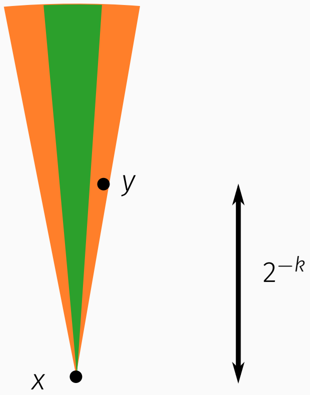
"Proof" of the key geometric lemma



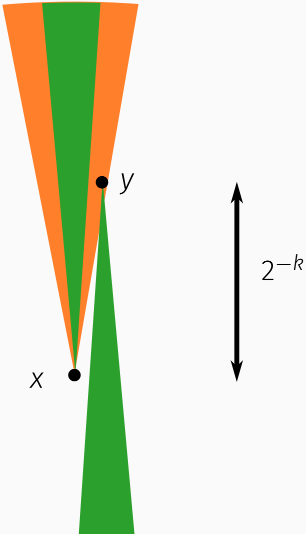
“Proof” of the key geometric lemma



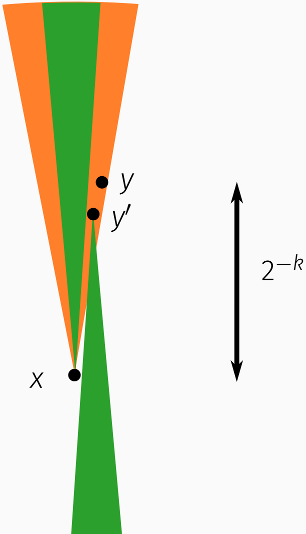
“Proof” of the key geometric lemma



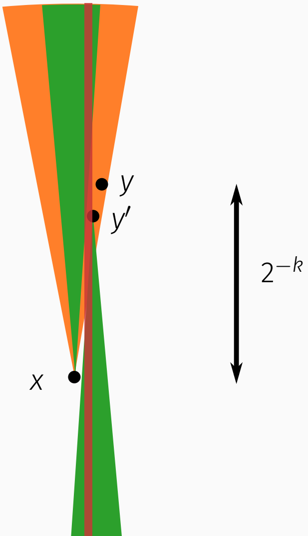
“Proof” of the key geometric lemma



“Proof” of the key geometric lemma



"Proof" of the key geometric lemma



Qualitative ULFL

Suppose that E is compact, and for every $x \in E$ we have

$$\liminf_{r \to 0} \frac{\operatorname{Fav}(E \cap B(x,r))}{r} > 0.$$

Does this imply $\gamma(E) > 0$?

A question

Qualitative ULFL

Suppose that E is compact, and for every $x \in E$ we have

$$\liminf_{r \to 0} \frac{\mathrm{Fav}(E \cap B(x,r))}{r} > 0.$$

Does this imply $\gamma(E) > 0$?

Thank you!