The visibility problem for Ahlfors regular sets

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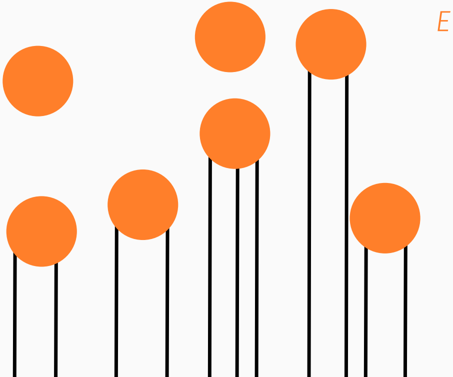
$$\mathrm{Vis}_\theta(E) = \left\{ x \in E \;:\; \ell_{x,\theta} \cap E = \{x\} \right\}$$



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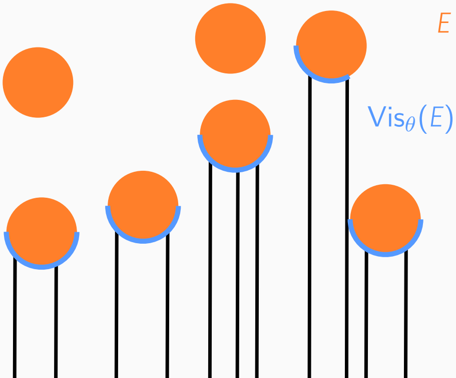

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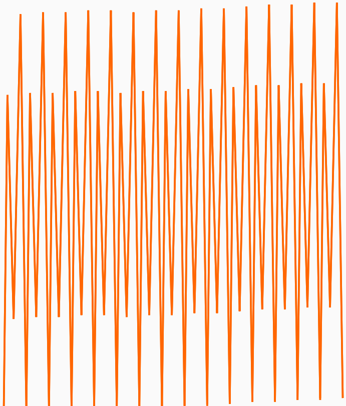

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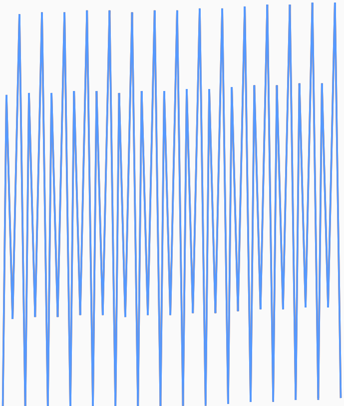

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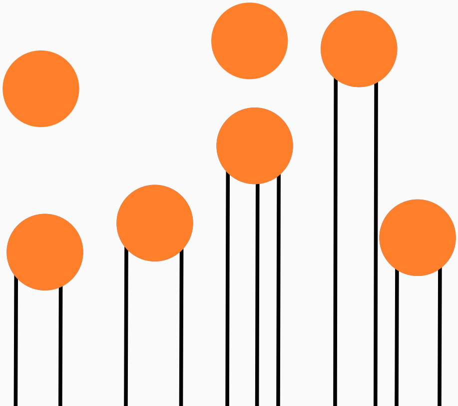

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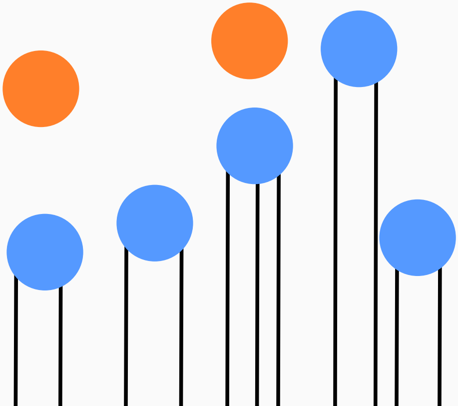

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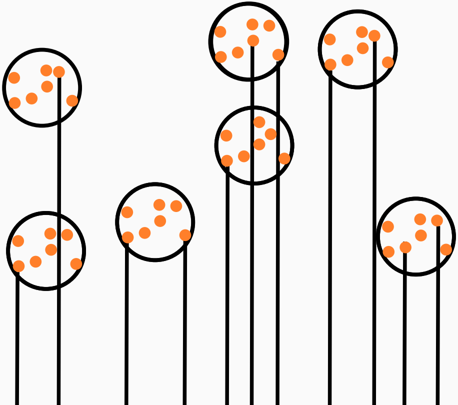

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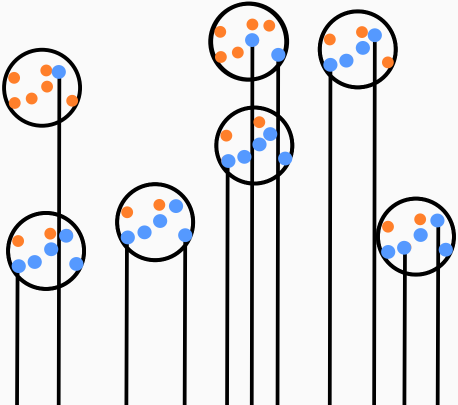

Past work

The conjecture has been verified for

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- **fractal percolation** (Arhosalo-Järvenpää-Järvenpää-Rams-Shmerkin '12)
- **self-similar and self-affine sets satisfying additional hypotheses** (JJMO '03, Falconer-Fraser '13, Rossi '21, Järvenpää-Järvenpää-Suomala-Wu '22)

Past work

Progress for general sets:

- (Järvenpää-Järvenpää-Niemela '04)

$$0 < \mathcal{H}^s(E) < \infty \quad \Rightarrow \quad \mathcal{H}^s(\mathrm{Vis}_\theta(E)) = 0$$

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- (Orponen ’22)

$$\dim \mathrm{Vis}_\theta(E) \leq 1.99$$

Still unknown:

$$\dim \mathrm{Vis}_\theta(E) \overset{?}{<} \dim E$$

Ahlfors regular sets

A compact set E is **s-Ahlfors regular** if for all $x \in E$, $0 < r < \text{diam}(E)$

$$\mathcal{H}^s(E \cap B(x, r)) \sim r^s.$$

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E.g. all self-similar sets satisfying the open set condition are Ahlfors regular.

New results

Theorem (D. '23)

If E is compact, then

$$\dim \mathrm{Vis}_\theta(E) \leq 2 - \frac{1}{6}.$$

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If E is s-Ahlfors regular, $s > 1$, then

$$\dim \mathrm{Vis}_\theta(E) \leq s - \alpha(s-1),$$

where $\alpha = 0.1835\ldots$

Orponen’s approach

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Fix $\theta \in \mathbb{S}^1$. We want to show

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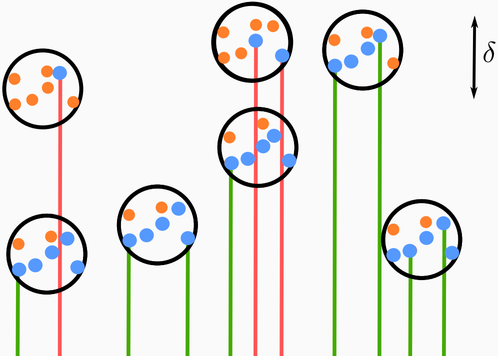
Fix $\delta > 0$. Let $\mathcal{L}$ be the lines with direction θ. We divide

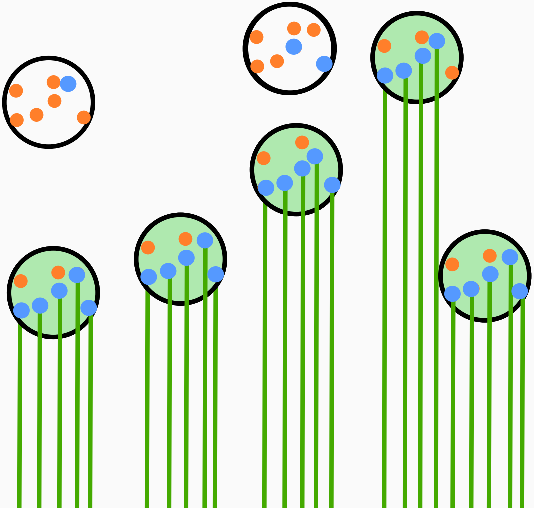
- good lines: $\ell \in \mathcal{L}_G$ if $\ell \cap E$ is similar to $\ell(\delta) \cap E(\delta)$
- bad lines: $\ell \in \mathcal{L}_B$ otherwise



Good part

Set $L_G := \bigcup_{\ell \in \mathcal{L}_G} \ell$ and $L_B := \bigcup_{\ell \in \mathcal{L}_B} \ell$.

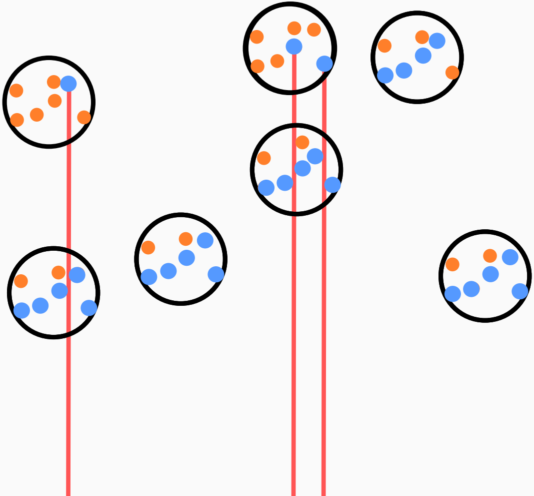
You can easily estimate $\mathcal{H}^{2-\tau}_{\infty}(\mathrm{Vis}_\theta(E) \cap L_G)$.



Bad part

To estimate $\mathrm{Vis}_\theta(E) \cap L_B$, one uses Fourier analysis to show that

$$\mathcal{H}^{1-\tau}_\infty(\pi_\theta(L_B)) = 0.$$



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All in all,

$$\mathcal{H}^{2-\tau}_\infty(\mathrm{Vis}_\theta(E)) \le \mathcal{H}^{2-\tau}_\infty(\mathrm{Vis}_\theta(E) \cap L_G) + \mathcal{H}^{2-\tau}_\infty(\mathrm{Vis}_\theta(E) \cap L_B) = 0,$$

and so $\dim \mathrm{Vis}_\theta(E) \le 2 - \tau$.

■

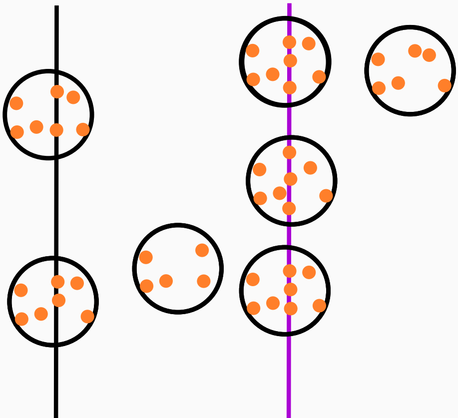
Improvement for Ahlfors regular sets

Slices of fractals

Theorem (Marstrand '54)

If $s > 1$ and $0 < \mathcal{H}^s(E) < \infty$, then for a.e. θ and $\mathcal{H}^s$-a.e. $x \in E$

$$\dim(E \cap \ell_{x,\theta}) = s - 1.$$



Heavy lines

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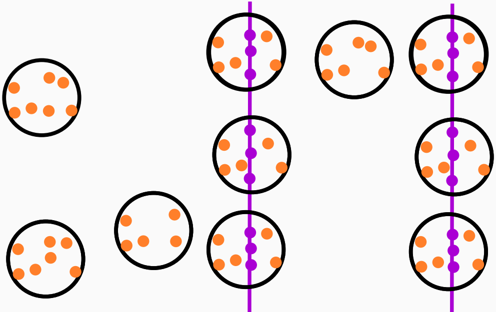
For $\theta \in \mathbb{S}^1$ we define the **heavy part** of E as

$$H_\theta(E) = E \cap \bigcup_{\ell \in \mathcal{H}_\theta} \ell,$$

where $\mathcal{H}_\theta$ is the collection of heavy lines for E with direction θ.



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Theorem (D. '23)

If E is s-Ahlfors regular, $s > 1$, then for a.e. θ

$$\dim H_\theta(E) \leq 1.$$

Compare with Marstrand: $\mathcal{H}^s(H_\theta(E)) = 0$.

Proof of the dimension drop

Theorem (D. '23)

If E is s-Ahlfors regular, $s > 1$, then

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Fix $\theta \in \mathbb{S}^1$. We want to show

$$\mathcal{H}^{s-\tau}_\infty(\mathrm{Vis}_\theta(E)) = 0$$

for any $\tau < \alpha(s-1)$.

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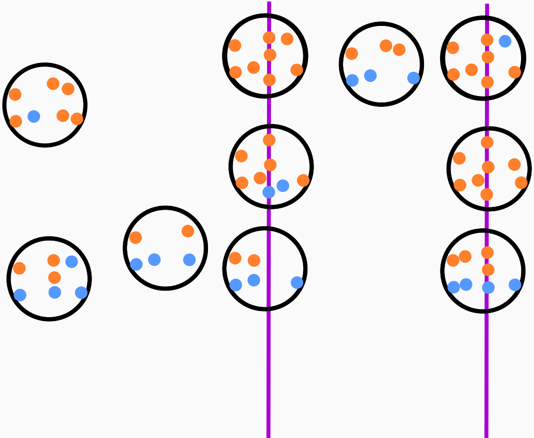
Fix $\delta > 0$. Let $\mathcal{L}$ be the lines with direction θ. We divide

- $\ell \in \mathcal{H}$ if ℓ is heavy for E



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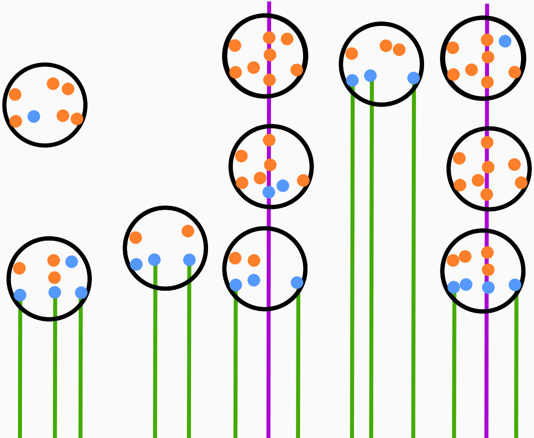

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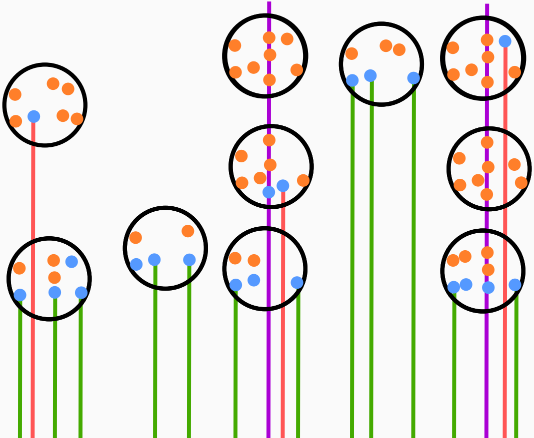

Conclusion

We estimate:

- the heavy part

$$\mathcal{H}^{s-\tau}_{\infty}(\mathsf{Vis}_{\theta}(E) \cap \bigcup_{\ell \in \mathcal{H}} \ell) \leq \mathcal{H}^{s-\tau}_{\infty}(H_{\theta}(E)) = 0,$$

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Hence, $\dim \mathrm{Vis}_\theta(E) \leq s - \tau$.

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- The estimate for dimension of heavy parts has been improved and generalized in [D.-Orponen-Wang '23]. Can this be used to get dimension drop for $\mathrm{Vis}_\theta(E)$ for general sets?

Thank you!