Favard length of random Cantor sets

Damian Dąbrowski (joint work with Alan Chang, Giacomo Del Nin)

Institute of Mathematics of the Polish Academy of Sciences (IMPAN)

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Outline

1. Motivation: removable sets for Lipschitz harmonic functions
2. Favard length problem
3. New results for some random Cantor sets

Previously on the Analysis and PDE seminar...

Carleson-type removability for p-parabolic equations

Michał Borowski[1] Theo Elenius[2] Leah Kurki Schätzler[3] David Stolnicki

[1]University of Warsaw [2]Aalto University [3]University of Coimbra

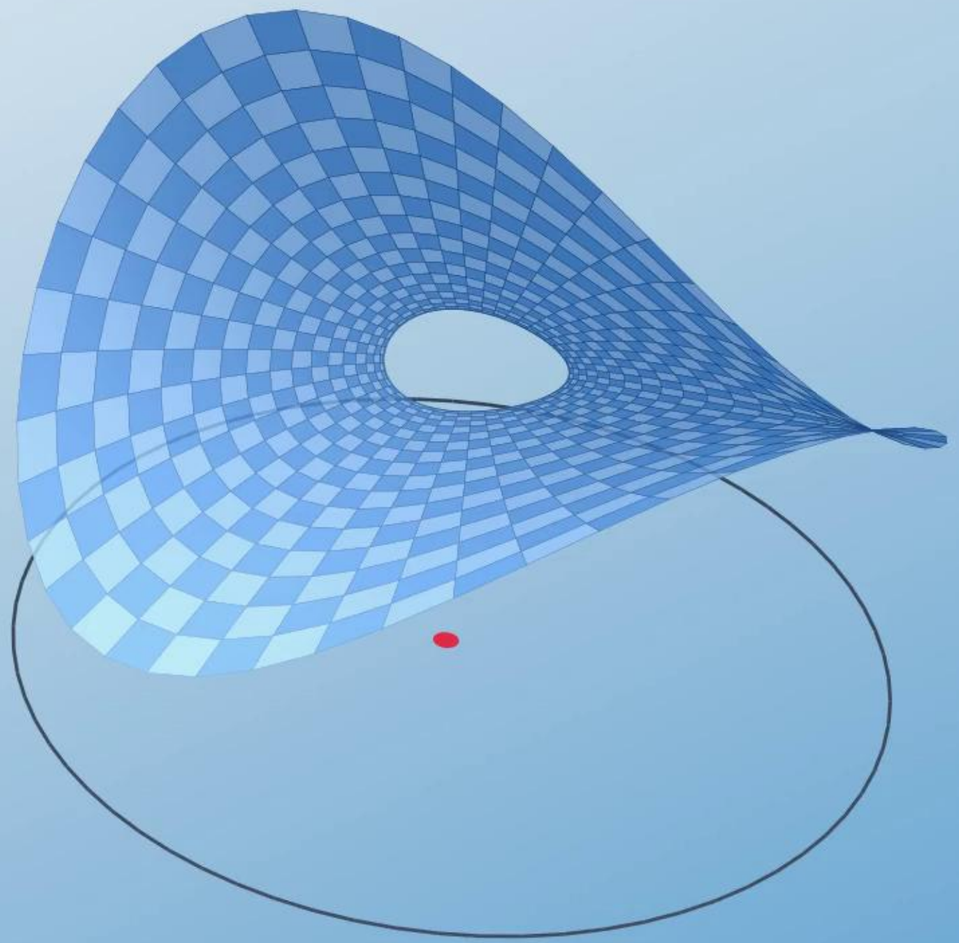
Analysis & PDE Seminar

19 November 2026

What is Carleson-type removability?

$$-\Delta u = 0 \quad \text{in } B \setminus \{0\} \subset \mathbb{R}^2$$

$$u = x^2 - y^2$$



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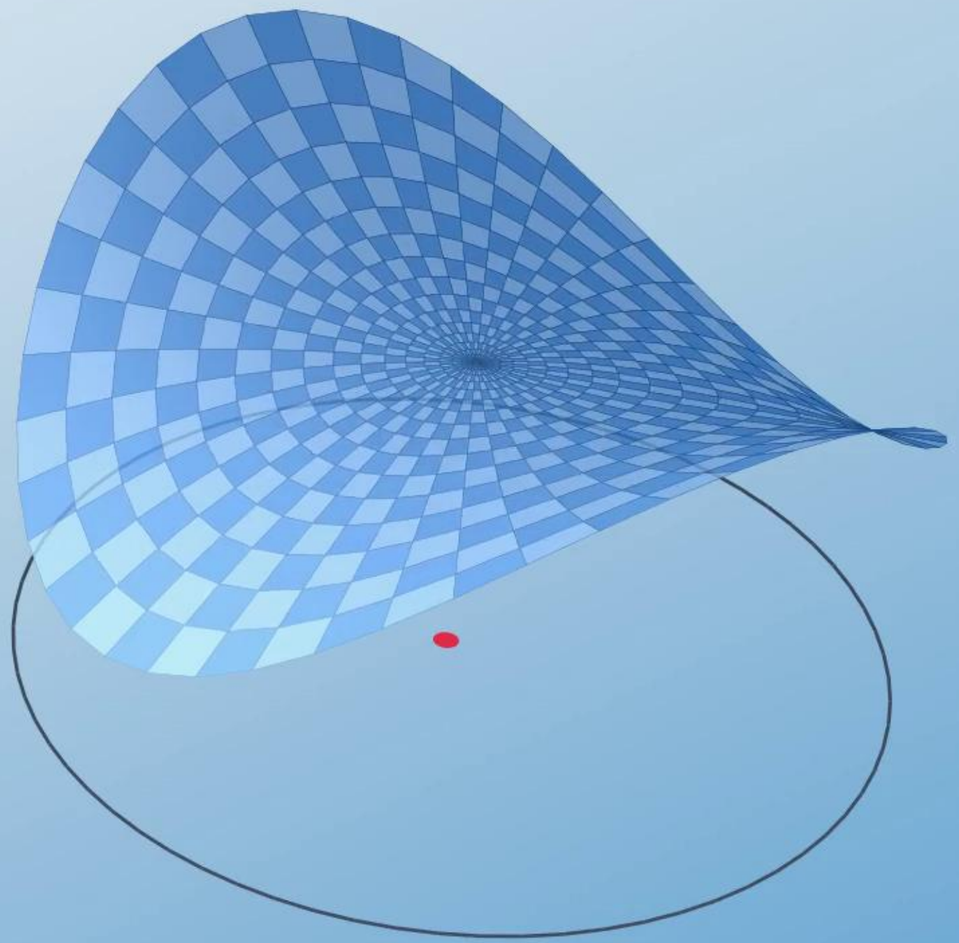

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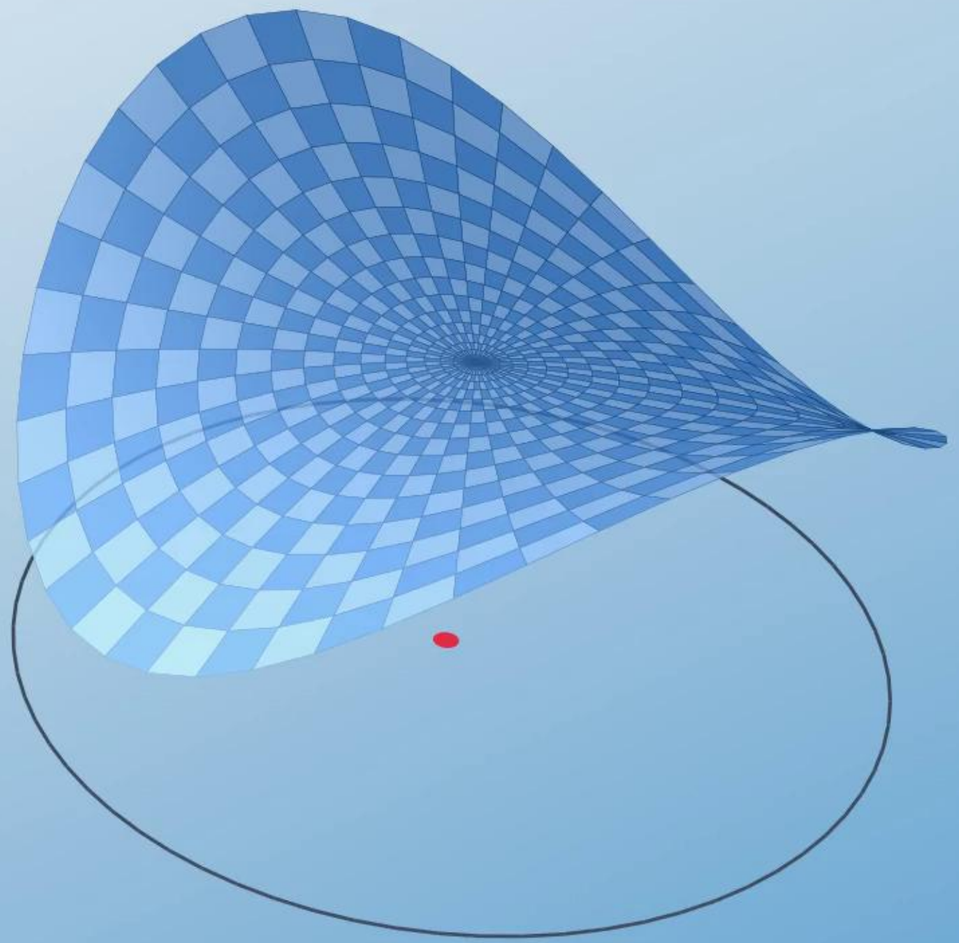

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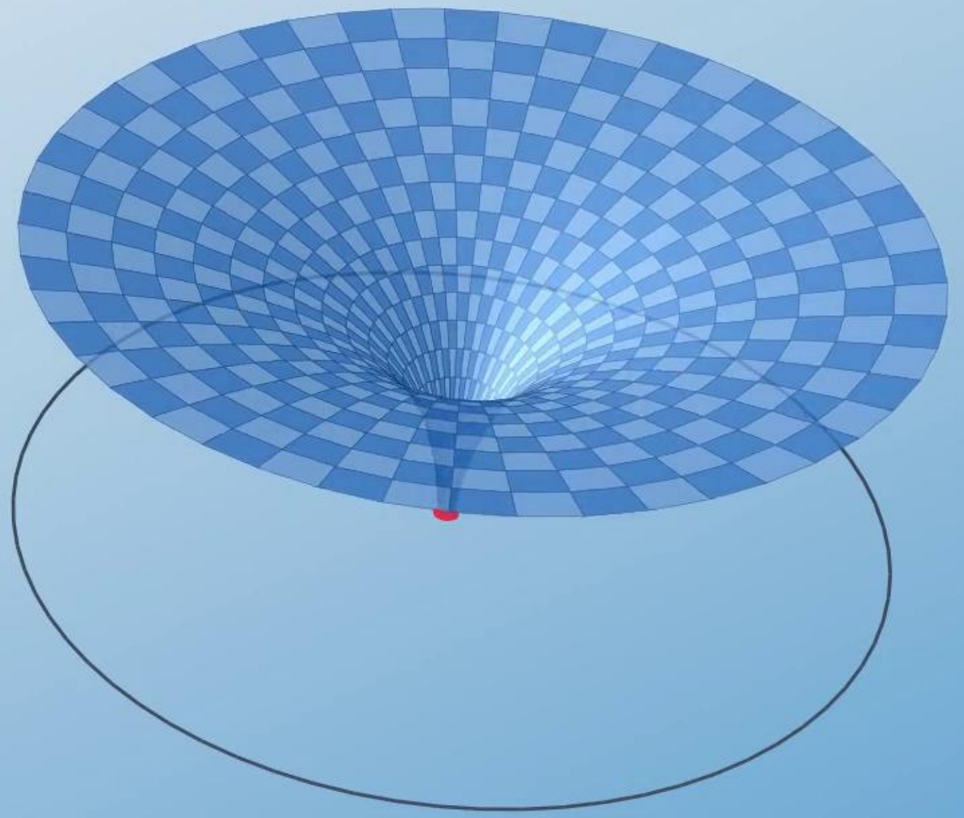
$u = \log|x|$

not removable: u is unbounded



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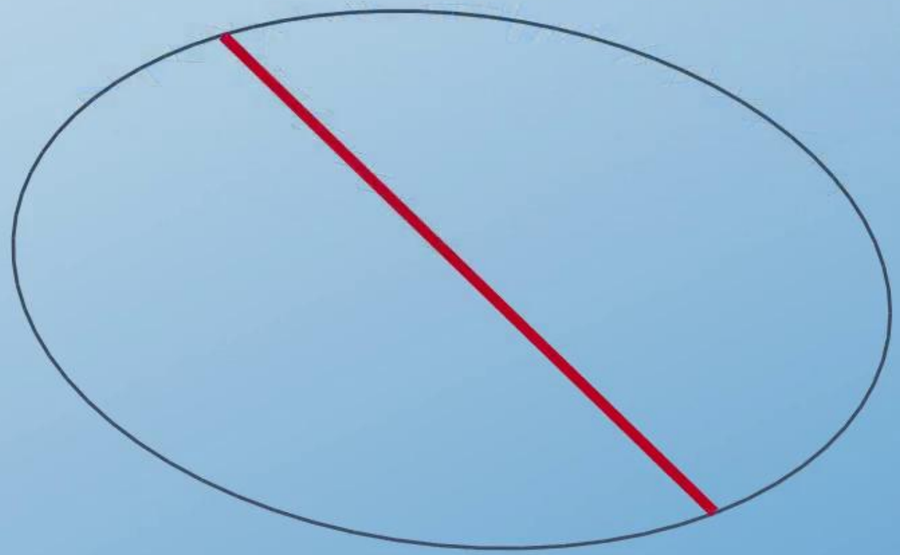

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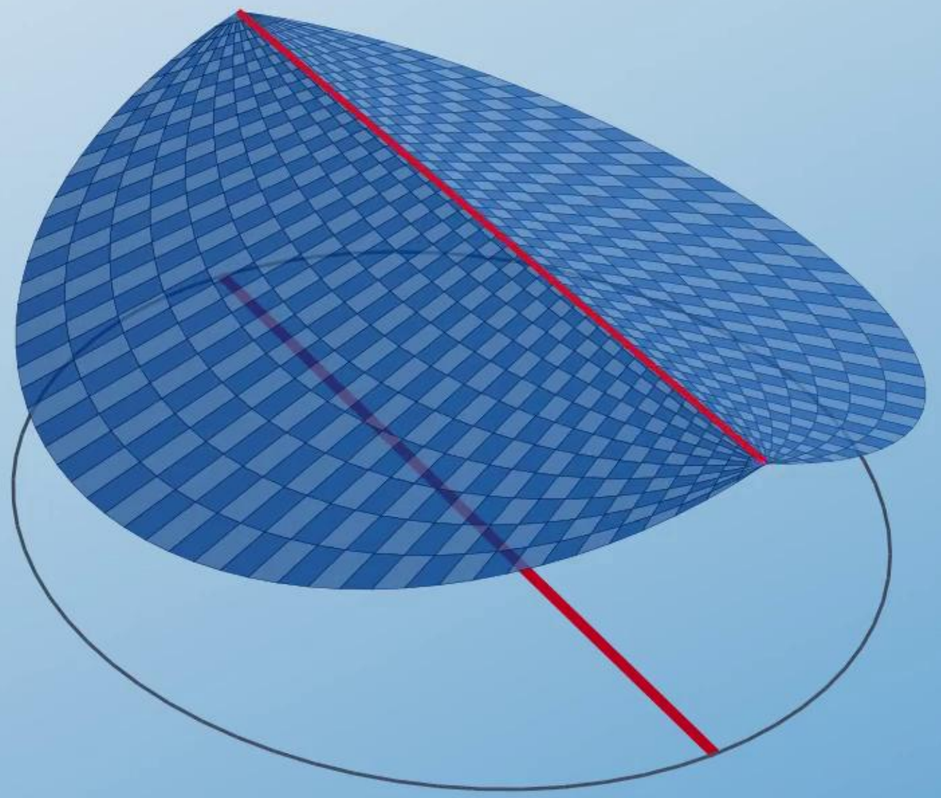
not removable: u is bounded, even Lipschitz, but not harmonic across E



What is Carleson-type removability?

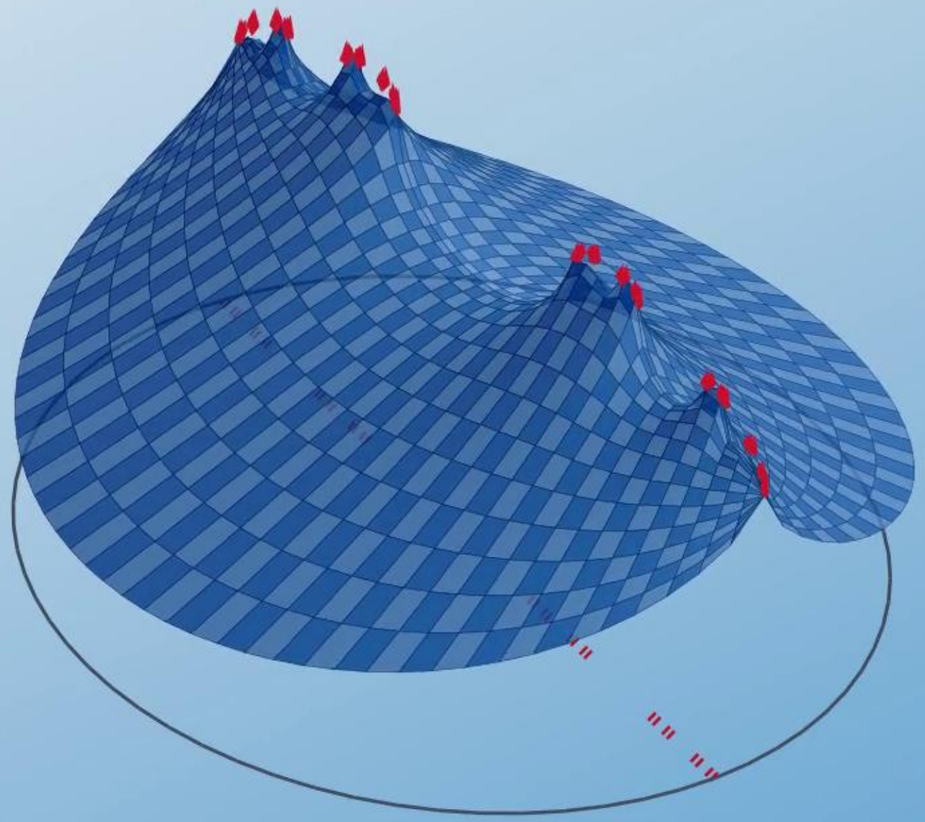
$-\Delta u = 0 \quad \text{in } B \setminus E, \qquad E = \text{Cantor set}$

$$u(z) = \int_E \log \frac{1}{|z-\xi|}\, d\mu(\xi)$$

μ: the Cantor measure on E

u is Hölder continuous up to E:
$u \in C^{0,\beta}(B), \quad \beta = \log 2 / \log 3$

not removable: $-\Delta u = 2\pi\mu \neq 0$ in B



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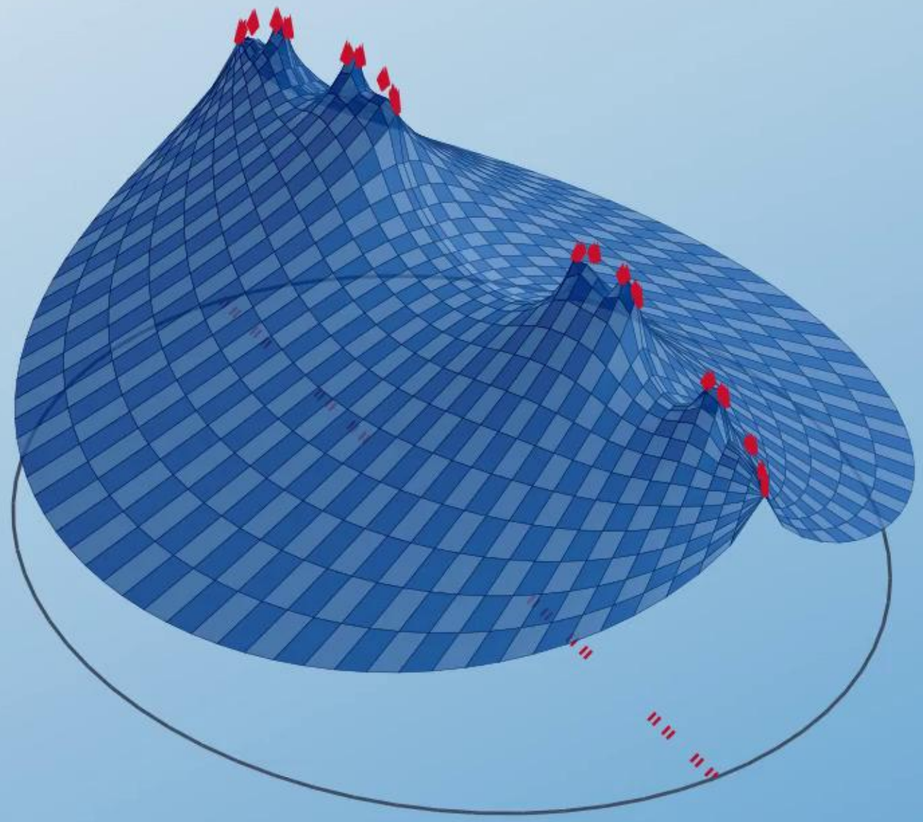
Carleson (1963)

$\Omega \subset \mathbb{R}^n$ open, $E \subset \Omega$ compact, $0 < \alpha < 1$:
E is removable for every $u \in C^{0,\alpha}(\Omega)$
$\iff \quad \mathcal{H}^{n-2+\alpha}(E) = 0$

L. Carleson, Math. Scand. 12 (1963), 15–18



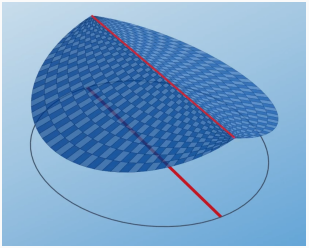
What happens for $\alpha = 1$?

Endpoint of Carleson's theorem:

$$E \subset \mathbb{R}^2 \text{ is } \textbf{removable for Lipschitz harmonic functions} \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$$

Example

A segment $I \subset \mathbb{R}^2$ is non-removable for Lipschitz harmonic functions.



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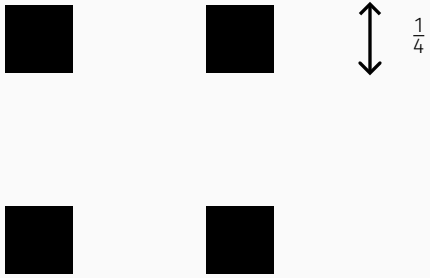
Example

A segment $I \subset \mathbb{R}^2$ is non-removable for Lipschitz harmonic functions.

Theorem (Vitushkin '59, Garnett, Ivanov '70s)

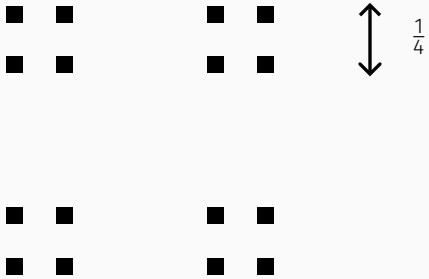
There exists a removable set $E \subset \mathbb{R}^2$ with $\mathcal{H}^1(E) > 0$.

Four-corners Cantor set



K_1

Four-corners Cantor set



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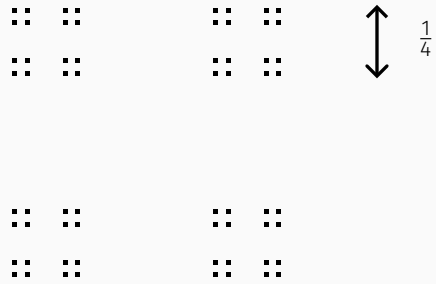

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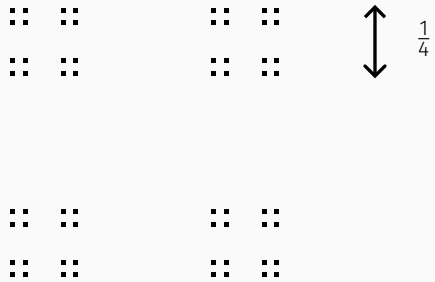
- $K = \bigcap_{n=1}^{\infty} K_n$ – four-corners Cantor set.

Four-corners Cantor set



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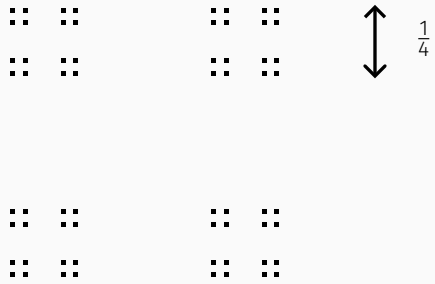

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- $K = \bigcap_{n=1}^{\infty} K_n$ – four-corners Cantor set.
- $0 < \mathcal{H}^1(K) < \infty$
- K is removable for Lipschitz harmonic functions [Garnett, Ivanov]

Vitushkin's conjecture

Two examples with $0 < \mathcal{H}^1(E) < \infty$:

- a segment is non-removable
- the four-corners Cantor set is removable

How to characterize removability for Lipschitz harmonic functions???

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$$E \subset \mathbb{R}^2 \text{ is removable} \quad \Leftrightarrow \quad \mathsf{Fav}(E) = 0$$

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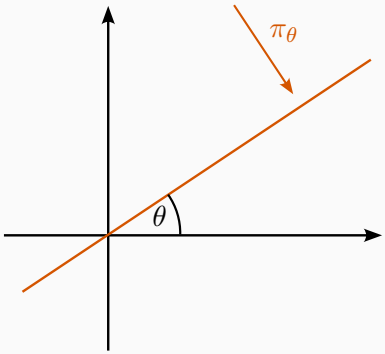
[1] for bounded analytic functions, but this is equivalent [Volberg 2003]

Favard length

Let $\pi_\theta(x) \coloneqq x \cdot (\cos\theta, \sin\theta)$.



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Favard length (or Buffon's needle probability) of a Borel set $E \subset \mathbb{R}^2$ is

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Vitushkin's conjecture (1967)

$$E \subset \mathbb{R}^2 \text{ is removable} \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0 \quad \Leftrightarrow \quad |\pi_\theta(E)| = 0 \quad \text{for a.e. } \theta$$

Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$E \subset \mathbb{R}^2 \text{ is removable} \quad \Leftrightarrow \quad \mathsf{Fav}(E) = 0$$

In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**!

Theorem (Calderón '77, David '98)

If $E \subset \mathbb{R}^2$ satisfies $\mathcal{H}^1(E) < \infty$, then

$$E \subset \mathbb{R}^2 \text{ is removable} \quad \Leftrightarrow \quad \mathsf{Fav}(E) = 0 \quad \Leftrightarrow \quad E \text{ is purely unrectifiable,}$$

i.e. $\mathcal{H}^1(E \cap \Gamma) = 0$ for every rectifiable curve.

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In the case $\mathcal{H}^1(E) = \infty$, Vitushkin's conjecture is **false**! [Mattila '86]

Theorem (Jones-Murai '88, Joyce-Mörters '00, Chang-D.-Del Nin '26)

There exists a non-removable set $E \subset \mathbb{R}^2$ with $\mathsf{Fav}(E) = 0$.

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There exists a non-removable set $E \subset \mathbb{R}^2$ with $\mathsf{Fav}(E) = 0$.

Thus, $\quad E \text{ is removable} \quad \not\Leftarrow \quad \mathsf{Fav}(E) = 0$

Open problem: $\quad E \text{ is removable} \quad \overset{?}{\Rightarrow} \quad \mathsf{Fav}(E) = 0$

Analytic capacity

Theorem (Ahlfors '48)

$$E \text{ is removable} \quad \Leftrightarrow \quad \gamma(E) = 0$$

$\gamma(E)$ is the analytic capacity of E:

$$\gamma(E) = \sup\{|f'(\infty)| \ : \ f : \mathbb{C} \setminus E \to \mathbb{C} \text{ analytic}, \ \|f\|_\infty \leq 1\},$$
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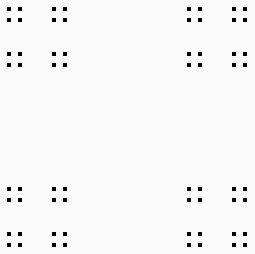
Equivalently: $\gamma(E) \geq C\,\mathrm{Fav}(E)?$

Example: four-corners Cantor set

$$\gamma(E) \geq C \operatorname{Fav}(E) \qquad (*)$$

Recall the four-corners Cantor set $K = \bigcap_{n=1}^{\infty} K_n$.



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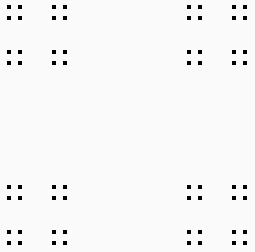

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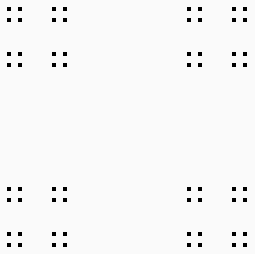

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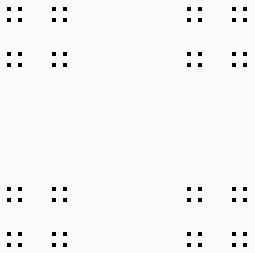

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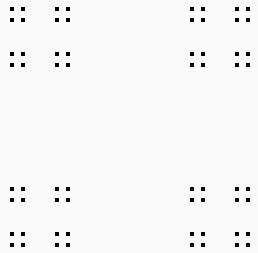
$\gamma(K_n) \sim n^{-1/2}$, but $\mathsf{Fav}(K_n)$ is not known...



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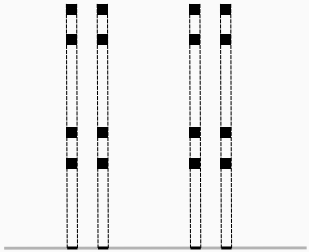

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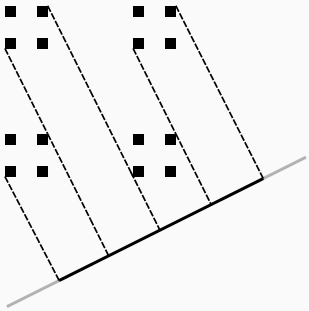

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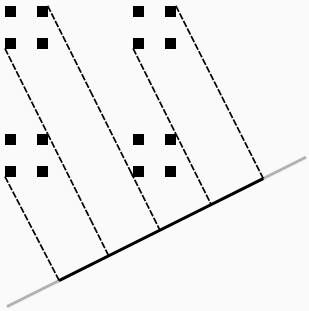

What happens on average???

Lower bounds

Theorem (Mattila '90)

For any compact $E \subset \mathbb{R}^2$

$$\mathsf{Fav}(E) \gtrsim \mathrm{Cap}_1(E),$$

where

$$\mathrm{Cap}_1(E) = \sup_{\mu,\, \operatorname{supp}\mu \subset E} \frac{1}{\mathcal{I}_1(\mu)}.$$

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Theorem (Bateman-Volberg '08)

This is not sharp:

$$\mathsf{Fav}(K_n) \gtrsim \frac{\log n}{n}.$$

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- $\mathsf{Fav}(E_n) \lesssim n^{-c}$ for some other self-similar sets [Bond-Volberg '10], [Łaba-Zhai '10], [Bond-Łaba-Volberg '11], [Łaba-Marshall '22], [Marshall '26+]

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- proofs: Fourier analysis, estimates for trigonometric polynomials

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- Peres-Solomyak '02: very slow decay
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- Marshall (Aug '26+): $\mathsf{Fav}(K_n) \lesssim_\varepsilon n^{-1/5+\varepsilon}$

- $\mathsf{Fav}(E_n) \lesssim n^{-c}$ for some other self-similar sets [Bond-Volberg '10], [Łaba-Zhai '10], [Bond-Łaba-Volberg '11], [Łaba-Marshall '22], [Marshall '26+]

- general self-similar sets with nice separation: only sub-polynomial bound

$$\mathsf{Fav}(E_n) \lesssim \exp(-c\sqrt{\log n}) \qquad \text{(Bond-Volberg '12)}$$

- proofs: Fourier analysis, estimates for trigonometric polynomials
- Vitushkin's conjecture $\Rightarrow$

$$\mathsf{Fav}(E_n) \lesssim \gamma(E_n) \sim n^{-1/2}.$$

We don't even know that for the four-corners Cantor set!

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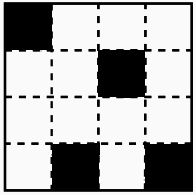
For many 1-dimensional grid-based random Cantor sets

$$\mathrm{Fav}(E_n) \sim \frac{1}{n}$$

almost surely and in expectation.



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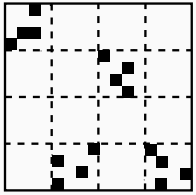

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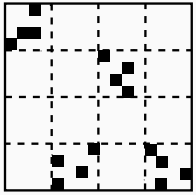
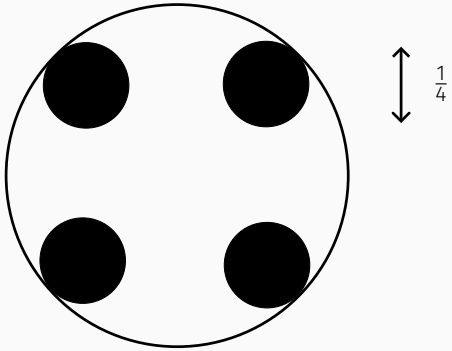


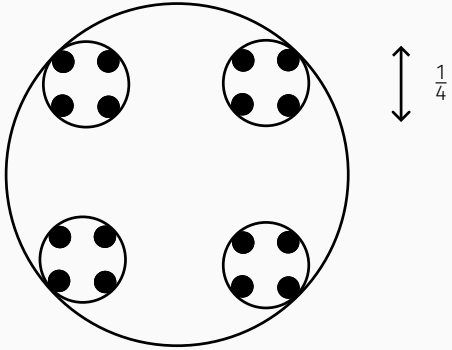
$\rightsquigarrow$ for such E_n Mattila's bound is sharp a.s.

Four-corners Cantor set revisited



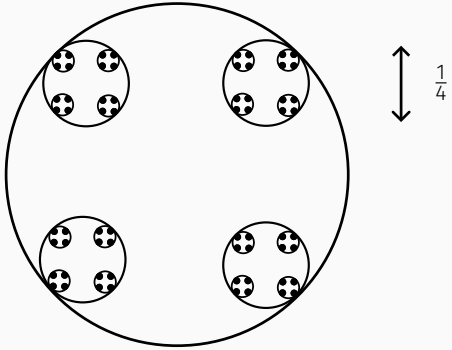
K_1

Four-corners Cantor set revisited



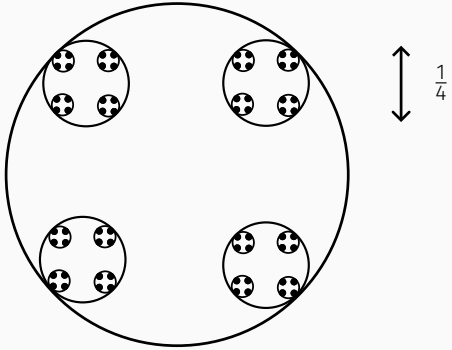
K_2

Four-corners Cantor set revisited



K_3

Four-corners Cantor set revisited



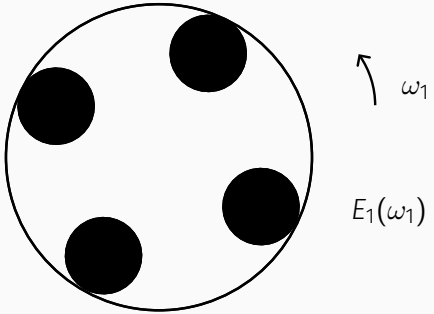
K_3

$K = \bigcap_n K_n$ – usual four corners Cantor set

Random Cantor sets – disc models

- “synchronized rotations model”

  $\rightsquigarrow$ $(\omega_n)_{n\geq 1}$ i.i.d. uniform random variables on $[0, 2\pi]$



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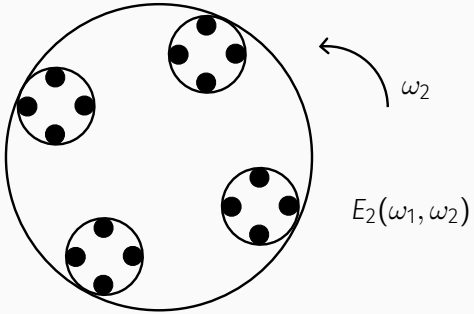

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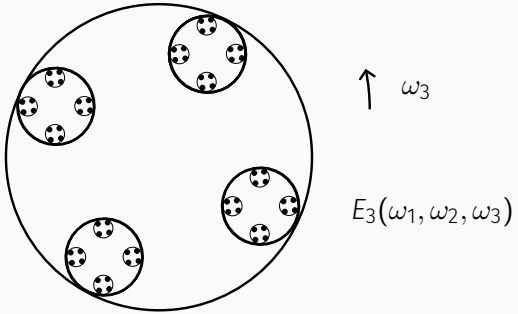

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- “independent rotations model”

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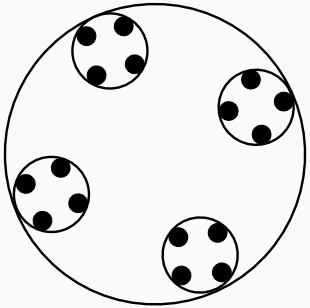

Random Cantor sets – disc models

Theorem (Zhang '20)

For the synchronized model

$$\mathbb{E}\,\mathrm{Fav}(E_n) \sim \frac{1}{n}.$$

Theorem (Vardakis-Volberg '24)

For the independent rotations model

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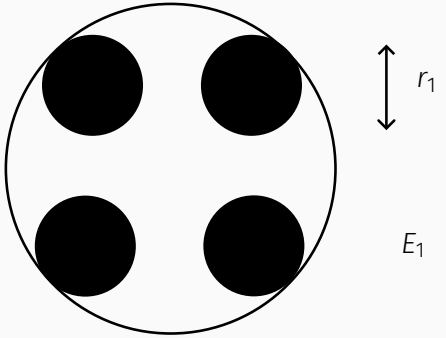
Non-homogeneous Cantor sets

Non-homogeneous four-corners set (Garnett 70s)

For $r_0 > r_1 > r_2 > \cdots > 0$ satisfying $r_{n+1} \leq r_n/3$ consider:



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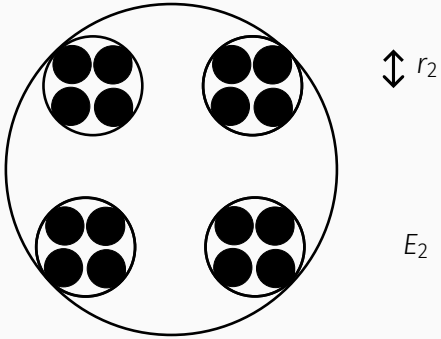

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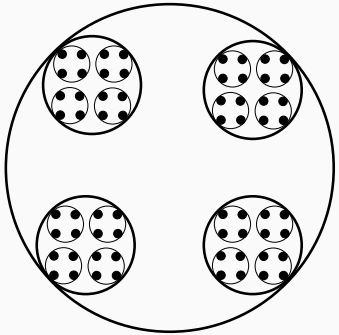

E_3

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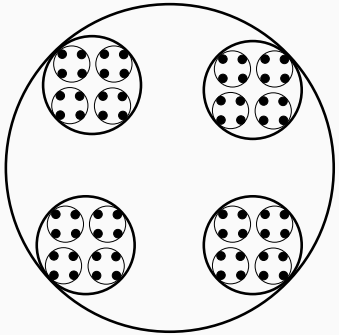

E_3

- $E = \bigcap_n E_n$ – non-homogeneous four-corners Cantor set
- $r_n = 4^{-k}$ $\quad\rightsquigarrow\quad$ usual four-corners Cantor set

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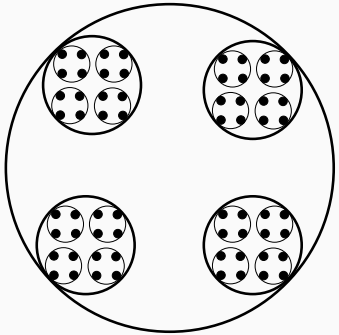

E_3

Problem (Mattila '04)

Characterize sequences $(r_n)_{k\in\mathbb{N}}$ such that $\mathsf{Fav}(E) = 0$.

What is known

Sufficient conditions:

- Besicovitch projection theorem:

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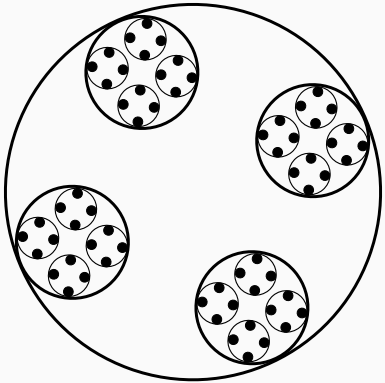
$$\mathsf{Fav}(E) = 0 \quad \Rightarrow \quad \mathrm{Cap}_1(E) = 0 \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} \frac{1}{r_n 4^n} = \infty.$$

Random variant

[Chang-D.-Del Nin '26] studies a random variant of these sets:

- "synchronized rotations model" $\rightsquigarrow$ $(\omega_n)_{n\geq 1}$ - i.i.d. uniform r.v. on $[0, 2\pi]$
- "independent rotations model" $\rightsquigarrow$ i.i.d. uniform r.v. ω_D for each disc D



Solution for the random variant

Theorem (Chang-D.-Del Nin '26)

For both models

$$\mathbb{E}\,\mathsf{Fav}(E_n) \sim \mathrm{Cap}_1(E_n) \sim \left(\sum_{k=0}^{n} \frac{1}{r_k 4^k}\right)^{-1}. \tag{1}$$

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- Corollary: $\gamma(E_n) \gtrsim \mathbb{E}\,\mathrm{Fav}(E_n)$
- in the independent rotations model we adapt [CSS '25] to get an a.s. version of (1).

Proof for synchronized rotations

Setup

- Fix $r_0 > r_1 > r_2 > \dots$
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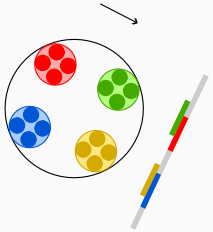

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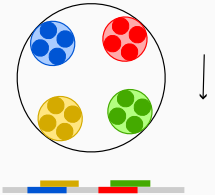

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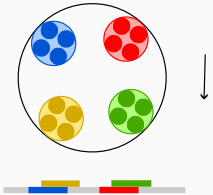

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Focus on the homogeneous case $r_n = 4^{-n}$:

$$\boxed{a_n \leq a_{n-1} - ca_{n-1}^2} \quad \rightsquigarrow \quad \mathbb{E}\,\mathsf{Fav}(E_n) \sim a_n \lesssim \frac{1}{n}$$

Proof of inductive inequality (Zhang '20)

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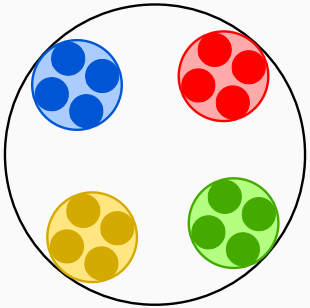
$$E_n(\omega_1, \ldots, \omega_n) = \bigcup_{j=1}^{4} \frac{1}{4} E_{n-1}(\omega_2, \ldots, \omega_n) + v_j(\omega_1)$$



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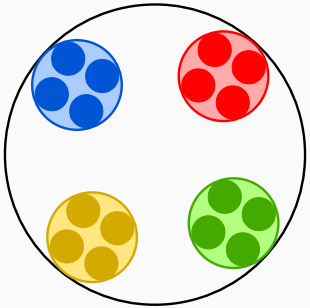

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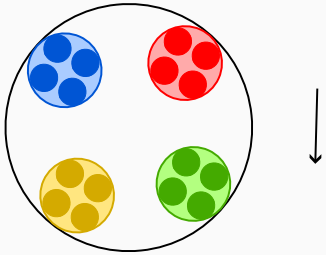
By the inclusion-exclusion principle

$$|\pi_0(E_n)| = \sum_{j=1}^{4} |\pi_0(S_j)| - \sum_{j \neq k} |\pi_0(S_j) \cap \pi_0(S_k)|.$$



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By the inclusion-exclusion principle

$$|\pi_0(E_n)| = \sum_{j=1}^{4} |\pi_0(S_j)| - \sum_{j \neq k} |\pi_0(S_j) \cap \pi_0(S_k)|.$$

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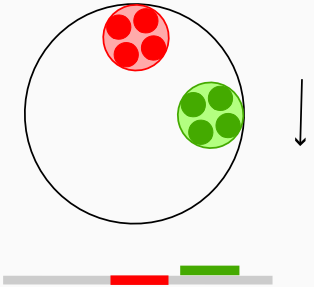

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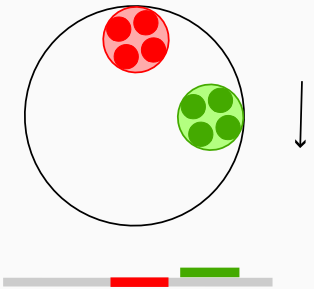

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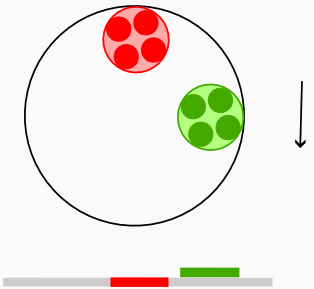

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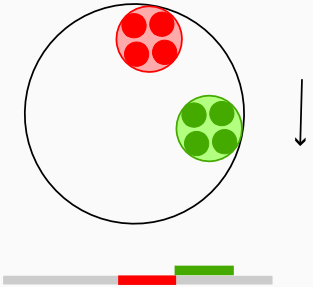

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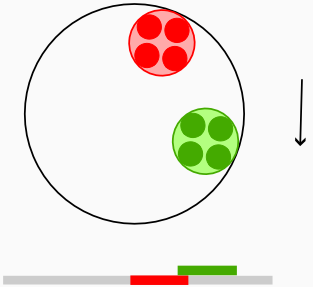

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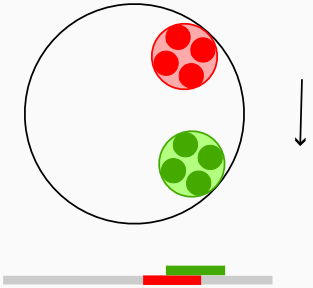

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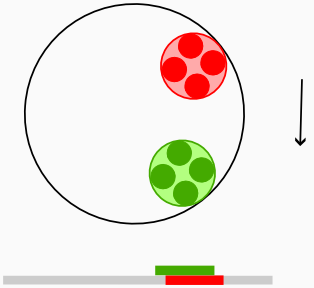

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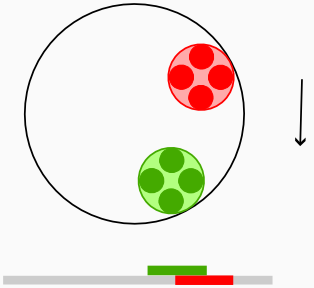

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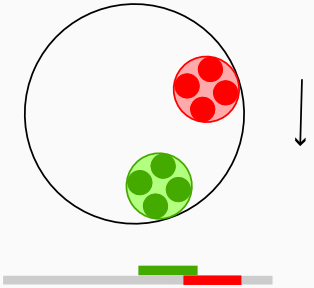

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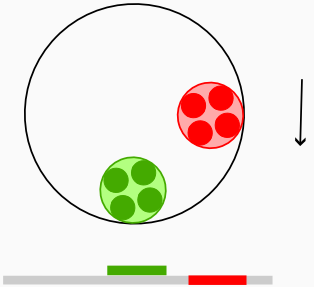

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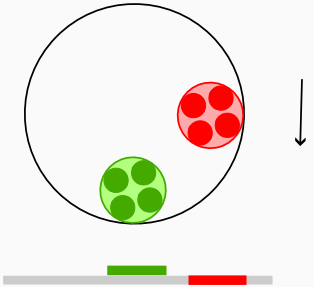

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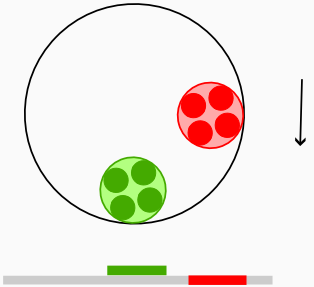

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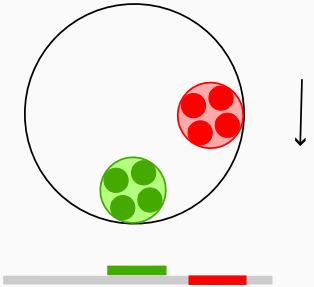

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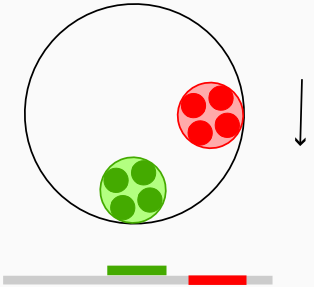

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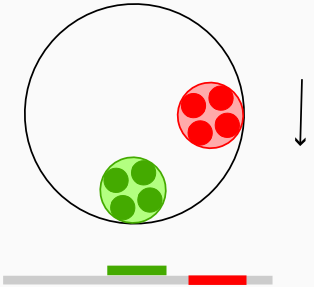

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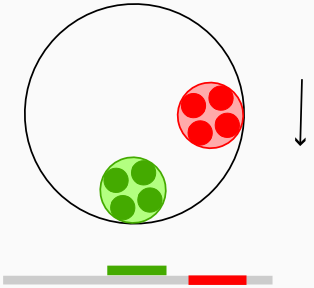

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Take $\mathbb{E}_{\omega_2,\ldots,\omega_n}$:

$$\mathbb{E}|\pi_0(S_j) \cap \pi_0(S_k)| \approx \mathbb{E}_{\omega_2,\ldots,\omega_n}|\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|^2$$

Size of the intersection

- $S_j = \frac{1}{4} E_{n-1}(\omega_2, \ldots, \omega_n) + v_j(\omega_1)$
- Goal: $\mathbb{E}|\pi_0(S_j) \cap \pi_0(S_k)| \geq c a_{n-1}^2$
- $\mathbb{E}_{\omega_1}|\pi_0(S_j) \cap \pi_0(S_k)| \approx |\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|^2$

Take $\mathbb{E}_{\omega_2,\ldots,\omega_n}$:

$$\begin{aligned}\mathbb{E}|\pi_0(S_j) \cap \pi_0(S_k)| &\approx \mathbb{E}_{\omega_2,\ldots,\omega_n}|\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|^2 \\ &\geq (\mathbb{E}_{\omega_2,\ldots,\omega_n}|\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|)^2\end{aligned}$$

Size of the intersection

- $S_j = \frac{1}{4}E_{n-1}(\omega_2, \ldots, \omega_n) + v_j(\omega_1)$
- Goal: $\mathbb{E}|\pi_0(S_j) \cap \pi_0(S_k)| \geq ca_{n-1}^2$
- $\mathbb{E}_{\omega_1}|\pi_0(S_j) \cap \pi_0(S_k)| \approx |\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|^2$

Take $\mathbb{E}_{\omega_2,\ldots,\omega_n}$:

$$\begin{aligned}\mathbb{E}|\pi_0(S_j) \cap \pi_0(S_k)| &\approx \mathbb{E}_{\omega_2,\ldots,\omega_n}|\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|^2 \\ &\geq (\mathbb{E}_{\omega_2,\ldots,\omega_n}|\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|)^2 = a_{n-1}^2.\end{aligned}$$

Size of the intersection

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$$\Rightarrow \quad a_n \leq a_{n-1} - ca_{n-1}^2 \quad \Rightarrow \quad a_n \lesssim \frac{1}{n}.$$

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Size of the intersection

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- Goal: $\mathbb{E}|\pi_0(S_j) \cap \pi_0(S_k)| \geq c a_{n-1}^2$
- $\mathbb{E}_{\omega_1}|\pi_0(S_j) \cap \pi_0(S_k)| \approx |\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|^2$

Take $\mathbb{E}_{\omega_2,\ldots,\omega_n}$:

$$\begin{aligned}\mathbb{E}|\pi_0(S_j) \cap \pi_0(S_k)| &\approx \mathbb{E}_{\omega_2,\ldots,\omega_n}|\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|^2 \\ &\geq (\mathbb{E}_{\omega_2,\ldots,\omega_n}|\pi_0(E_{n-1}(\omega_2, \ldots, \omega_n))|)^2 = a_{n-1}^2.\end{aligned}$$

$$\Rightarrow \quad a_n \leq a_{n-1} - c a_{n-1}^2 \quad \Rightarrow \quad a_n \lesssim \frac{1}{n}.$$

■

Thank you!