The geometry of singular integral operators

Damian Dąbrowski

Institute of Mathematics of the Polish Academy of Sciences (IMPAN)

Funded by
the European Union

European Research Council
Established by the European Commission

Big picture

partial differential equations

harmonic analysis

geometric measure theory

Big picture

partial differential equations $\leftrightsquigarrow$ harmonic analysis

quantitative rectifiability

geometric measure theory

Singular integral operators

Singular integral operators

Given a kernel $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ we say that the operator

$$Tf(x) := \int_{\mathbb{R}} K(x-y)\, f(y)\, dy$$

is a **singular integral operator** if $K \notin L^1(\mathbb{R})$.

Singular integral operators

Given a kernel $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ we say that the operator

$$Tf(x) := \int_{\mathbb{R}} K(x-y)\, f(y)\, dy$$

is a **singular integral operator** if $K \notin L^1(\mathbb{R})$.

The prototypical example: the **Hilbert transform**

$$\mathcal{H}f(x) := \int_{\mathbb{R}} \frac{1}{x-y}\, f(y)\, dy$$

Singular integral operators

Given a kernel $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ we say that the operator

$$Tf(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y)\, f(y)\, dy$$

is a **singular integral operator** if $K \notin L^1(\mathbb{R})$.

The prototypical example: the **Hilbert transform**

$$\mathcal{H}f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)\, dy$$
$$= p.v. \int \frac{1}{x-y} f(y)\, dy$$

Singular integral operators

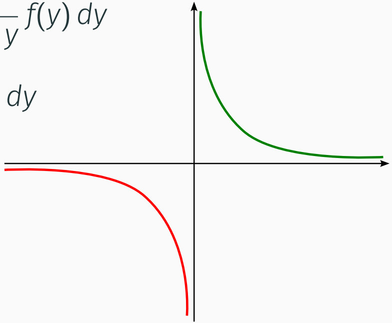
Given a kernel $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ we say that the operator

$$Tf(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y)\, f(y)\, dy$$

is a **singular integral operator** if $K \notin L^1(\mathbb{R})$.

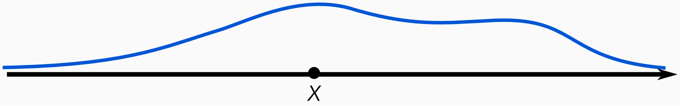
The prototypical example: the **Hilbert transform**

$$\mathcal{H}f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)\, dy$$
$$= p.v. \int \frac{1}{x-y} f(y)\, dy$$



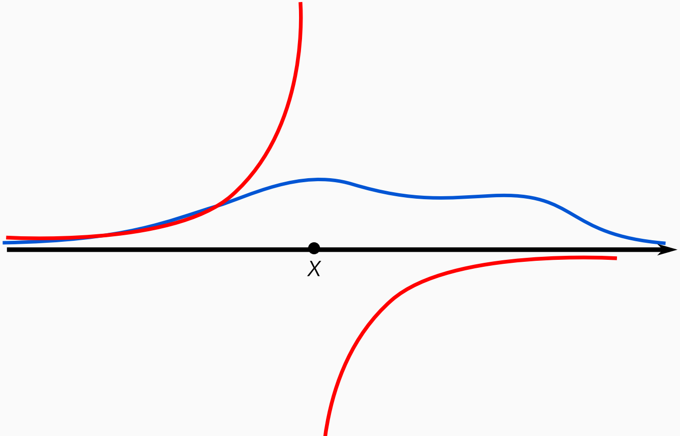
Cancellation in the Hilbert transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$



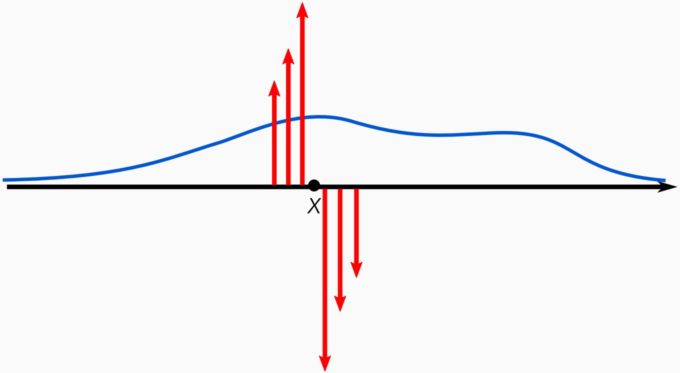
Cancellation in the Hilbert transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$



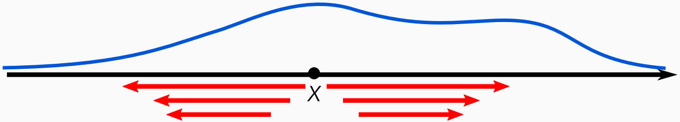
Cancellation in the Hilbert transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$



Cancellation in the Hilbert transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$



A little history

Problem

When can we define $\mathcal{H}f$ for $f \in L^p(\mathbb{R})$, $1 < p < \infty$? When is the Hilbert transform bounded on L^p, i.e.

$$\|\mathcal{H}f\|_{L^p} \leq C\|f\|_{L^p}?$$

A little history

Problem

When can we define $\mathcal{H}f$ for $f \in L^p(\mathbb{R})$, $1 < p < \infty$? When is the Hilbert transform bounded on L^p, i.e.

$$\|\mathcal{H}f\|_{L^p} \leq C\|f\|_{L^p}?$$

- the Hilbert transform is bounded on $L^2(\mathbb{R})$ [Hilbert 1900s]

A little history

Problem

When can we define $\mathcal{H}f$ for $f \in L^p(\mathbb{R})$, $1 < p < \infty$? When is the Hilbert transform bounded on L^p, i.e.

$$\|\mathcal{H}f\|_{L^p} \leq C\|f\|_{L^p}?$$

- the Hilbert transform is bounded on $L^2(\mathbb{R})$ **[Hilbert 1900s]**
- the Hilbert transform is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ **[Riesz 1928]**

A little history

Problem

When can we define $\mathcal{H}f$ for $f \in L^p(\mathbb{R})$, $1 < p < \infty$? When is the Hilbert transform bounded on L^p, i.e.

$$\|\mathcal{H}f\|_{L^p} \leq C\|f\|_{L^p}?$$

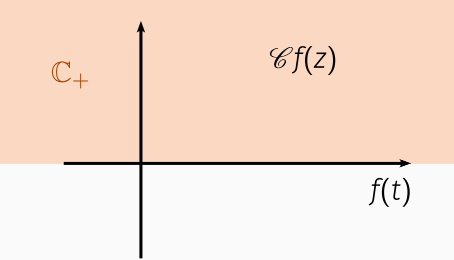
- the Hilbert transform is bounded on $L^2(\mathbb{R})$ **[Hilbert 1900s]**
- the Hilbert transform is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ **[Riesz 1928]**
- general theory of singular integral operators was developed by **Calderón** and **Zygmund** in the 1950s

Motivation: boundary values of analytic functions

Given $f \in C_c^\infty(\mathbb{R})$ consider the Cauchy integral

$$\mathscr{C}f(z) := c \int_{\mathbb{R}} \frac{f(t)}{z-t}\, dt,$$

$z \in \mathbb{C}_+ = \{z = x + iy \,:\, y > 0\}$.

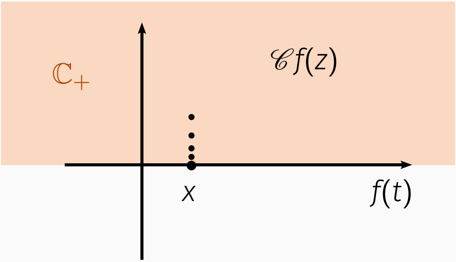


Motivation: boundary values of analytic functions

Given $f \in C_c^\infty(\mathbb{R})$ consider the Cauchy integral

$$\mathscr{C}f(z) := c \int_{\mathbb{R}} \frac{f(t)}{z-t}\, dt,$$

$z \in \mathbb{C}_+ = \{z = x + iy \;:\; y > 0\}.$



The function $\mathscr{C}_f(z)$ is holomorphic in $\mathbb{C}_+$, and

$$\mathscr{C}f(x+iy) \xrightarrow{y \to 0^+} f(x) + i\mathcal{H}f(x).$$

Studying boundary values of $\mathscr{C}f(z)$ for $f \in L^p(\mathbb{R})$ leads to questions on the L^p-boundedness of $\mathcal{H}$.

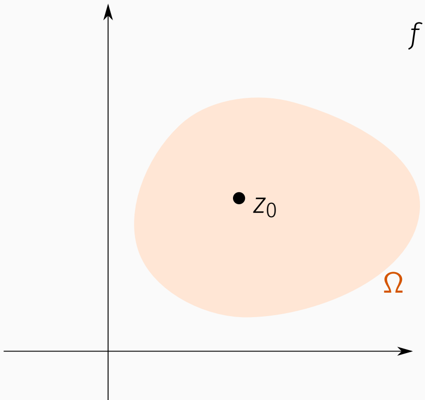
Painlevé problem

Riemann's theorem on removable singularities

Theorem (Riemann)

If $z_0 \in \Omega \subset \mathbb{C}$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is analytic and bounded, then f can by extended analytically to all of Ω.

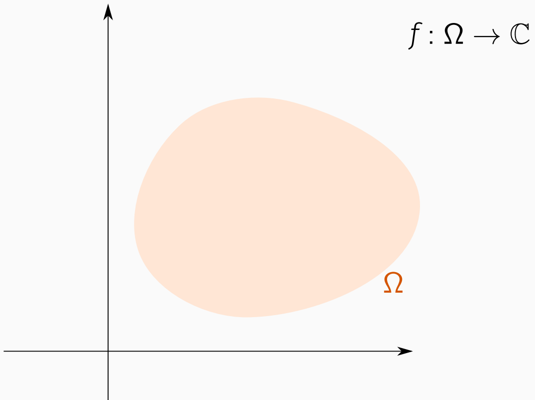
$f : \Omega \setminus \{z_0\} \to \mathbb{C}$



Riemann's theorem on removable singularities

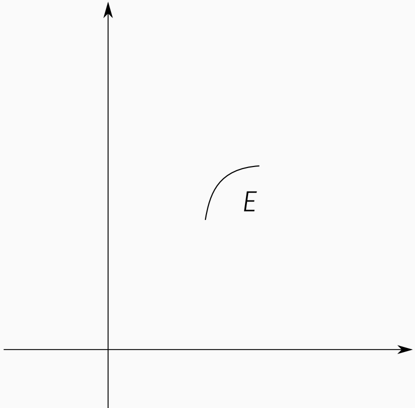
Theorem (Riemann)

If $z_0 \in \Omega \subset \mathbb{C}$ and $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is analytic and bounded, then f can by extended analytically to all of Ω.



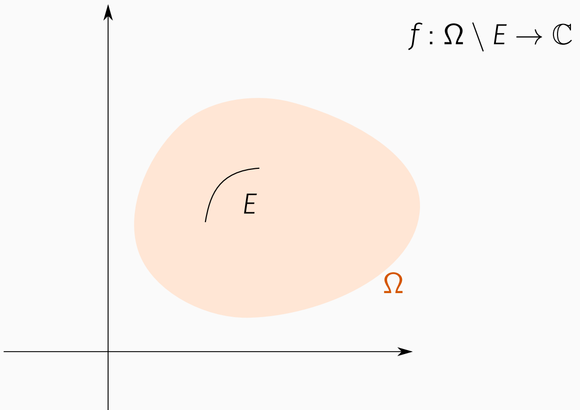
Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



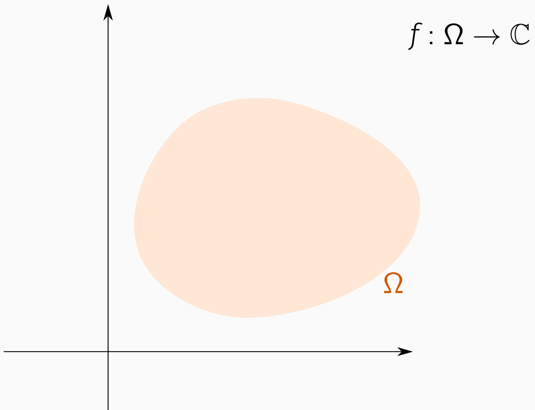
Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.

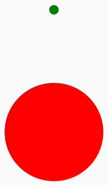


Examples

- a singleton is removable,

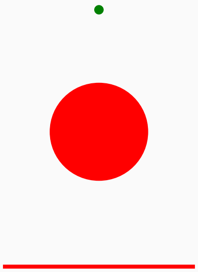
Examples

- a singleton is removable,
- a ball is non-removable,



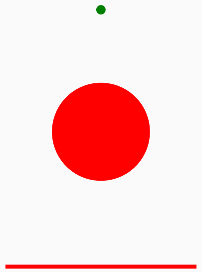
Examples

- a singleton is removable,
- a ball is non-removable,
- a segment is non-removable.



Examples

- a singleton is removable,
- a ball is non-removable,
- a segment is non-removable.



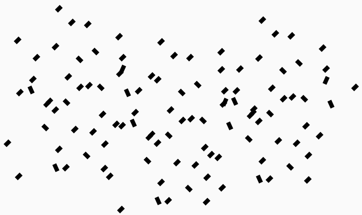
Painlevé problem

Find a geometric characterization of compact sets removable for bounded analytic functions.

Some classical results

- if $\mathrm{length}(E) = 0$, then E is removable (Painlevé 1892)
- if $\dim(E) > 1$, then E is non-removable
- if E is connected, then E is non-removable

So the Painlevé problem is concerned with 1-dimensional totally disconnected sets of positive length.



Some classical results

- if $\mathrm{length}(E) = 0$, then E is removable (Painlevé 1892)
- if $\dim(E) > 1$, then E is non-removable
- if E is connected, then E is non-removable

So the Painlevé problem is concerned with 1-dimensional totally disconnected sets of positive length.

Perhaps

$$E \text{ is removable} \quad \Leftrightarrow \quad \mathrm{length}(E) = 0?$$

Some classical results

- if $\text{length}(E) = 0$, then E is **removable** (Painlevé 1892)
- if $\dim(E) > 1$, then E is **non-removable**
- if E is connected, then E is **non-removable**

So the Painlevé problem is concerned with 1-dimensional totally disconnected sets of positive length.

Perhaps

$$\sout{E \text{ is removable} \quad \Leftrightarrow \quad \text{length}(E) = 0?}$$

No!

Removable set with positive length

Theorem (Vitushkin, Ivanov, Garnett 60s)

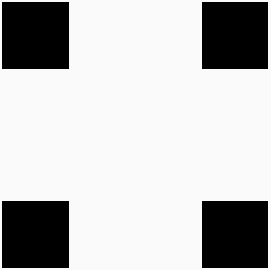
There exists a removable set $K \subset \mathbb{C}$ with $\operatorname{length}(K) > 0$.

Removable set with positive length

Theorem (Vitushkin, Ivanov, Garnett 60s)

There exists a removable set $K \subset \mathbb{C}$ with $\text{length}(K) > 0$.

The example of Ivanov and Garnett is the **4-corners Cantor set**:



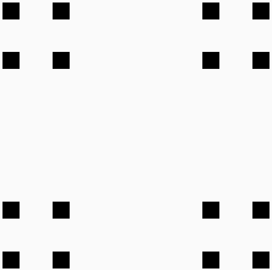
K_1

Removable set with positive length

Theorem (Vitushkin, Ivanov, Garnett 60s)

There exists a removable set $K \subset \mathbb{C}$ with $\text{length}(K) > 0$.

The example of Ivanov and Garnett is the **4-corners Cantor set**:



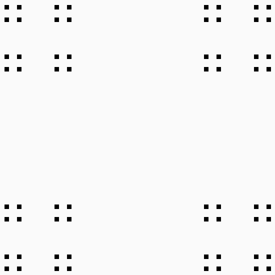
K_2

Removable set with positive length

Theorem (Vitushkin, Ivanov, Garnett 60s)

There exists a removable set $K \subset \mathbb{C}$ with $\text{length}(K) > 0$.

The example of Ivanov and Garnett is the **4-corners Cantor set**:



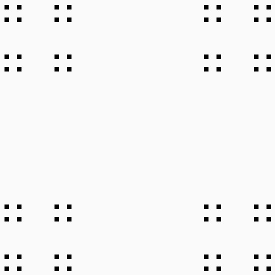
K_3

Removable set with positive length

Theorem (Vitushkin, Ivanov, Garnett 60s)

There exists a removable set $K \subset \mathbb{C}$ with $\mathrm{length}(K) > 0$.

The example of Ivanov and Garnett is the **4-corners Cantor set**:



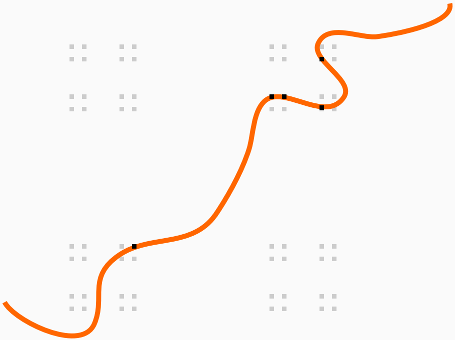
$$K = \bigcap_n K_n$$

Removable set with positive length

Theorem (Vitushkin, Ivanov, Garnett 60s)

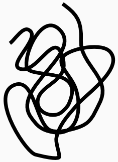
There exists a removable set $K \subset \mathbb{C}$ with $\text{length}(K) > 0$.

The example of Ivanov and Garnett is the 4-corners Cantor set:



Geometric measure theory

A **rectifiable curve** $\Gamma \subset \mathbb{R}^2$ is a curve with $\mathrm{length}(\Gamma) < \infty$.

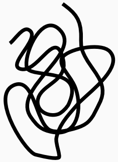


Rectifiable vs purely unrectifiable

A **rectifiable curve** $\Gamma \subset \mathbb{R}^2$ is a curve with $\mathrm{length}(\Gamma) < \infty$.

A set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of rectifiable curves Γ_i such that

$$\mathrm{length}\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$



Rectifiable vs purely unrectifiable

A **rectifiable curve** $\Gamma \subset \mathbb{R}^2$ is a curve with $\mathrm{length}(\Gamma) < \infty$.

A set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of rectifiable curves Γ_i such that

$$\mathrm{length}\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

We say that $F \subset \mathbb{R}^2$ is **purely unrectifiable** if for every rectifiable curve Γ

$$\mathrm{length}(F \cap \Gamma) = 0.$$

Conjectures of Denjoy and Vitushkin

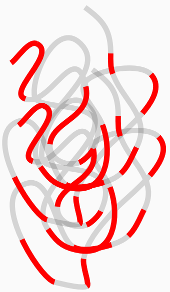
Theorem (Denjoy 1909)

If E is rectifiable and $\text{length}(E) > 0$, then E is non-removable.

Conjectures of Denjoy and Vitushkin

~~Theorem (Denjoy 1909)~~ Denjoy's conjecture

If E is rectifiable and $\operatorname{length}(E) > 0$, then E is non-removable.



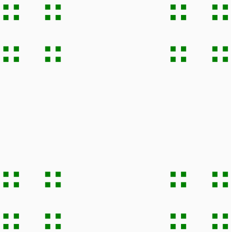
Conjectures of Denjoy and Vitushkin

~~Theorem (Denjoy 1909)~~ Denjoy's conjecture

If E is rectifiable and $\mathrm{length}(E) > 0$, then E is non-removable.

Vitushkin's conjecture (1967)

If E is purely unrectifiable and $\mathrm{length}(E) < \infty$, then E is removable



Conjectures of Denjoy and Vitushkin

~~Theorem (Denjoy 1909)~~ Denjoy's conjecture

If E is rectifiable and $\text{length}(E) > 0$, then E is non-removable.

Vitushkin's conjecture (1967)

If E is purely unrectifiable and $\text{length}(E) < \infty$, then E is removable

Both conjectures are true, solving the Painlevé problem for sets of finite length!

$$E \text{ is removable} \quad \Leftrightarrow \quad E \text{ is purely unrectifiable}$$

Key tool: the Cauchy integral

Recall the Cauchy integral we saw before:

$$\mathscr{C}f(z) := c \int_{\mathbb{R}} \frac{f(t)}{z-t}\, dt, \qquad z \in \mathbb{C}_+$$

Key tool: the Cauchy integral

Recall the Cauchy integral we saw before:

$$\mathscr{C}f(z) := c\int_{\mathbb{R}} \frac{f(t)}{z-t}\,dt, \qquad z \in \mathbb{C}_+$$

We can define similarly the **Cauchy integral of a measure** μ on $\mathbb{C}$:

$$\mathscr{C}\mu(z) = c\int \frac{1}{z-w}\,d\mu(w), \qquad z \in \mathbb{C} \setminus \operatorname{supp}\mu.$$

Key tool: the Cauchy integral

Recall the Cauchy integral we saw before:

$$\mathscr{C}f(z) := c \int_{\mathbb{R}} \frac{f(t)}{z-t}\, dt, \qquad z \in \mathbb{C}_+$$

We can define similarly the **Cauchy integral of a measure** μ on $\mathbb{C}$:

$$\mathscr{C}\mu(z) = c \int \frac{1}{z-w}\, d\mu(w), \qquad z \in \mathbb{C} \setminus \operatorname{supp} \mu.$$

We have

$$\bar{\partial}(\mathscr{C}\mu) = -\mu,$$

so that $\mathscr{C}\mu$ is holomorphic on $\mathbb{C} \setminus \operatorname{supp} \mu$, but not on $\operatorname{supp} \mu$.

Key tool: the Cauchy integral

Recall the Cauchy integral we saw before:

$$\mathscr{C}f(z) := c\int_{\mathbb{R}} \frac{f(t)}{z-t}\,dt, \qquad z \in \mathbb{C}_+$$

We can define similarly the **Cauchy integral of a measure** μ on $\mathbb{C}$:

$$\mathscr{C}\mu(z) = c\int \frac{1}{z-w}\,d\mu(w), \qquad z \in \mathbb{C} \setminus \operatorname{supp}\mu.$$

We have

$$\bar{\partial}(\mathscr{C}\mu) = -\mu,$$

so that $\mathscr{C}\mu$ is holomorphic on $\mathbb{C} \setminus \operatorname{supp}\mu$, but not on $\operatorname{supp}\mu$.

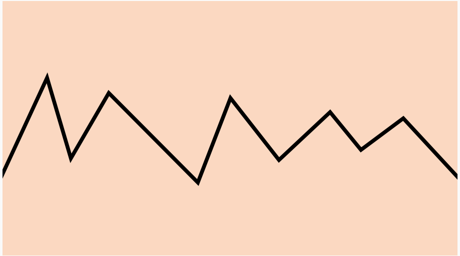
An approach to proving non-removability

Given $E \subset \mathbb{C}$, if we find a measure μ on E such that $\mathscr{C}\mu$ is bounded on $\mathbb{C} \setminus E$, then we get that E is non-removable!

Cauchy transform on a Lipschitz graph

Let $A : \mathbb{R} \to \mathbb{R}$ be Lipschitz, so that $|A(t) - A(s)| \leq C|t - s|$.

Set $\Gamma = \{t + iA(t) \ : t \in \mathbb{R}\}$.



Cauchy transform on a Lipschitz graph

Let $A : \mathbb{R} \to \mathbb{R}$ be Lipschitz, so that $|A(t) - A(s)| \leq C|t - s|$.

Set $\Gamma = \{t + iA(t) \ : t \in \mathbb{R}\}$.

For $f \in C_c^\infty(\Gamma)$ we consider the Cauchy integral

$$\mathscr{C}_\Gamma f(z) = c \int_\Gamma \frac{1}{z - w} f(w)\, dw, \qquad z \in \mathbb{C} \setminus \Gamma.$$

Cauchy transform on a Lipschitz graph

Let $A : \mathbb{R} \to \mathbb{R}$ be Lipschitz, so that $|A(t) - A(s)| \leq C|t - s|$.

Set $\Gamma = \{t + iA(t) \ : t \in \mathbb{R}\}$.

For $f \in C_c^\infty(\Gamma)$ we consider the Cauchy integral

$$\mathscr{C}_\Gamma f(z) = c \int_\Gamma \frac{1}{z - w} f(w)\, dw, \qquad z \in \mathbb{C} \setminus \Gamma.$$

For $z \in \Gamma$

$$\mathscr{C}_\Gamma f(z + i\delta) \xrightarrow{\delta \to 0^+} f(z) + i\mathcal{C}_\Gamma f(z),$$

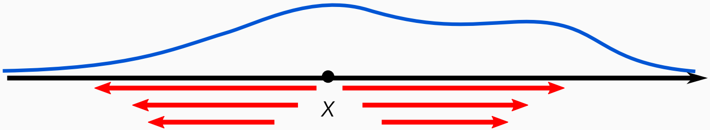
$$\mathscr{C}_\Gamma f(z - i\delta) \xrightarrow{\delta \to 0^+} f(z) - i\mathcal{C}_\Gamma f(z),$$

where $\mathcal{C}_\Gamma f(z)$ is **the Cauchy transform of f on Γ**

$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z - w} f(w)\, dw, \qquad z \in \Gamma.$$

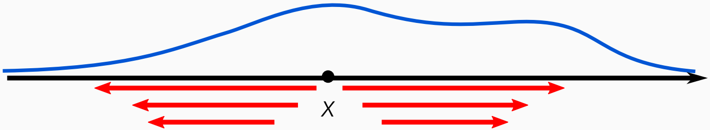
Cancellation in the Cauchy transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$

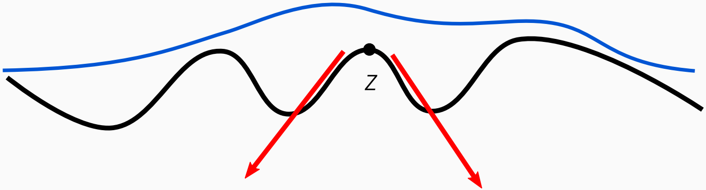


Cancellation in the Cauchy transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$

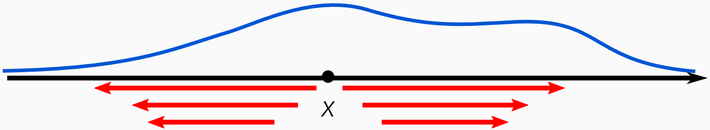


$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw$$

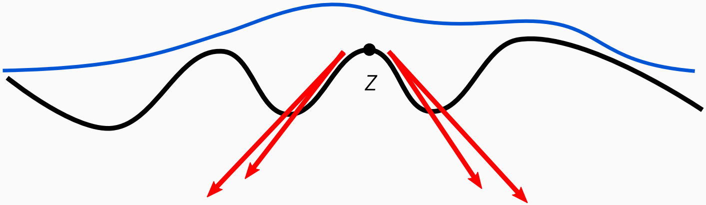


Cancellation in the Cauchy transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$

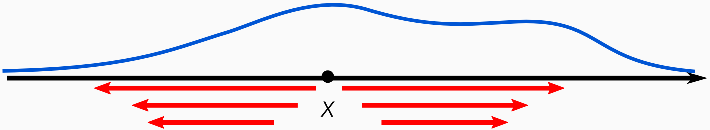


$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw$$

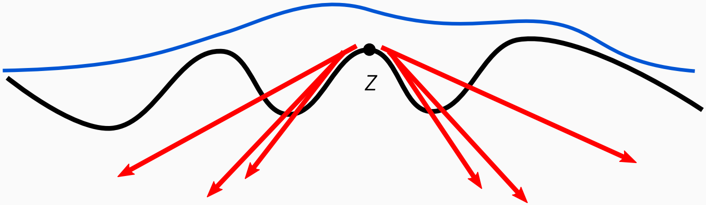


Cancellation in the Cauchy transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$

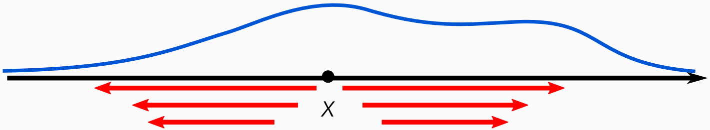


$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw$$

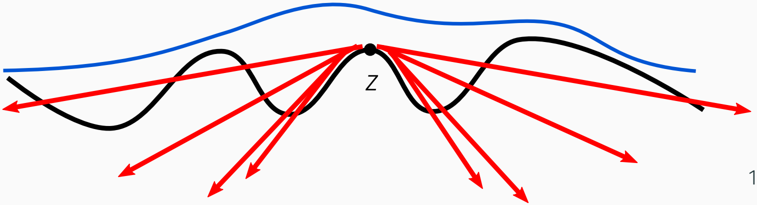


Cancellation in the Cauchy transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$

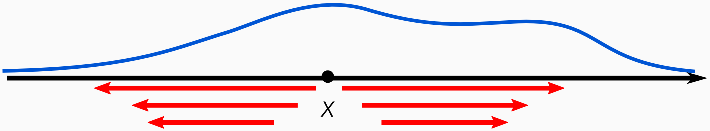


$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw$$

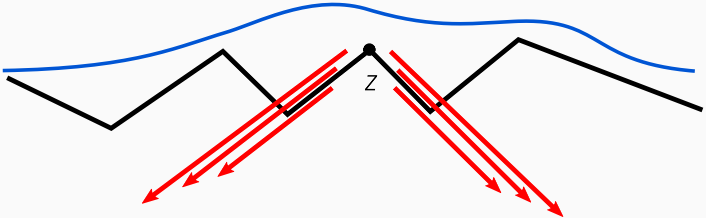


Cancellation in the Cauchy transform

$$\mathcal{H}f(x) = p.v. \int \frac{1}{x-y} f(y)\, dy$$



$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw$$



The more flatness, the more cancellations!

Solution to Denjoy's conjecture

$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw, \qquad z \in \Gamma.$$

Solution to Denjoy's conjecture

$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw, \qquad z \in \Gamma.$$

Theorem (Calderón 1977)

If $\mathrm{Lip}(\Gamma) \ll 1$, then

$$\|\mathcal{C}_\Gamma f\|_{L^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}.$$

Solution to Denjoy's conjecture

$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw, \qquad z \in \Gamma.$$

Theorem (Calderón 1977)

If $\mathrm{Lip}(\Gamma) \ll 1$, then

$$\|\mathcal{C}_\Gamma f\|_{L^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}.$$

Corollary (Marshall)

Denjoy's conjecture is true: if E is rectifiable and $\mathrm{length}(E) > 0$, then E is non-removable.

Solution to Denjoy's conjecture

$$\mathcal{C}_\Gamma f(z) = p.v. \int_\Gamma \frac{1}{z-w} f(w)\, dw, \qquad z \in \Gamma.$$

Theorem (Calderón 1977)

If $\mathrm{Lip}(\Gamma) \ll 1$, then

$$\|\mathcal{C}_\Gamma f\|_{L^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}.$$

Corollary (Marshall)

Denjoy's conjecture is true: if E is rectifiable and $\mathrm{length}(E) > 0$, then E is non-removable.

Theorem (Coifman-McIntosh-Meyer 1982)

$\mathrm{Lip}(\Gamma) < \infty$ is enough for $\|\mathcal{C}_\Gamma f\|_{L^2(\Gamma)} \leq C\|f\|_{L^2(\Gamma)}$.

Cauchy transform and removability

Given a measure μ on $\mathbb{C}$ and $f \in L^1_{loc}(\mu)$ we may consider **the Cauchy transform defined by μ**

$$\mathcal{C}_\mu f(z) = p.v. \int \frac{1}{z-w} f(w)\, d\mu(w), \qquad z \in \operatorname{supp} \mu.$$

Cauchy transform and removability

Given a measure μ on $\mathbb{C}$ and $f \in L^1_{loc}(\mu)$ we may consider **the Cauchy transform defined by μ**

$$\mathcal{C}_\mu f(z) = p.v. \int \frac{1}{z-w} f(w)\, d\mu(w), \qquad z \in \operatorname{supp} \mu.$$

Theorem (Davie-Øksendal, Murai 80s)

If E supports a measure μ such that $\mathcal{C}_\mu : L^2(\mu) \to L^2(\mu)$, then E is non-removable.

Cauchy transform and removability

Given a measure μ on $\mathbb{C}$ and $f \in L^1_{loc}(\mu)$ we may consider **the Cauchy transform defined by μ**

$$\mathcal{C}_\mu f(z) = p.v. \int \frac{1}{z-w} f(w)\, d\mu(w), \qquad z \in \operatorname{supp} \mu.$$

Theorem (Davie-Øksendal, Murai 80s)

If E supports a measure μ such that $\mathcal{C}_\mu : L^2(\mu) \to L^2(\mu)$, then E is non-removable.

Theorem (Tolsa 2003)

The converse holds: if E is non-removable, then it supports a measure μ such that $\mathcal{C}_\mu : L^2(\mu) \to L^2(\mu)$.

Cauchy transform and removability

Given a measure μ on $\mathbb{C}$ and $f \in L^1_{loc}(\mu)$ we may consider **the Cauchy transform defined by μ**

$$\mathcal{C}_\mu f(z) = p.v. \int \frac{1}{z-w} f(w)\, d\mu(w), \qquad z \in \operatorname{supp} \mu.$$

Theorem (Davie-Øksendal, Murai 80s)

If E supports a measure μ such that $\mathcal{C}_\mu : L^2(\mu) \to L^2(\mu)$, then E is non-removable.

Theorem (Tolsa 2003)

The converse holds: if E is non-removable, then it supports a measure μ such that $\mathcal{C}_\mu : L^2(\mu) \to L^2(\mu)$.

Problem

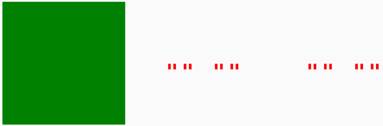
Which measures define L^2-bounded Cauchy transform?

Examples

- if μ is "more than 1-dimensional", then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓

Examples

- if μ is "more than 1-dimensional", then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is "less than 1-dimensional", then $\mathcal{C}_\mu$ is <u>not</u> $L^2(\mu)$-bounded ✗



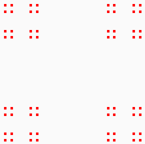
Examples

- if μ is "more than 1-dimensional", then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is "less than 1-dimensional", then $\mathcal{C}_\mu$ is not $L^2(\mu)$-bounded ✗
- if μ is the arc-length on a Lipschitz graph, then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓

Examples

- if μ is "more than 1-dimensional", then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is "less than 1-dimensional", then $\mathcal{C}_\mu$ is <u>not</u> $L^2(\mu)$-bounded ✗
- if μ is the arc-length on a Lipschitz graph, then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is the natural measure on the 4-corners Cantor set, then $\mathcal{C}_\mu$ is <u>not</u> $L^2(\mu)$-bounded ✗



Examples

- if μ is "more than 1-dimensional", then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is "less than 1-dimensional", then $\mathcal{C}_\mu$ is <u>not</u> $L^2(\mu)$-bounded ✗
- if μ is the arc-length on a Lipschitz graph, then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is the natural measure on the 4-corners Cantor set, then $\mathcal{C}_\mu$ is <u>not</u> $L^2(\mu)$-bounded ✗

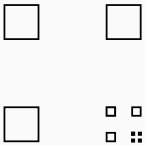
Can we characterize boundedness of Cauchy transform using rectifiability?

Examples

- if μ is "more than 1-dimensional", then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is "less than 1-dimensional", then $\mathcal{C}_\mu$ is <u>not</u> $L^2(\mu)$-bounded ✗
- if μ is the arc-length on a Lipschitz graph, then $\mathcal{C}_\mu$ is $L^2(\mu)$-bounded ✓
- if μ is the natural measure on the 4-corners Cantor set, then $\mathcal{C}_\mu$ is <u>not</u> $L^2(\mu)$-bounded ✗

Can we characterize boundedness of Cauchy transform using rectifiability? No ✗

- there exist rectifiable sets defining unbounded Cauchy transform

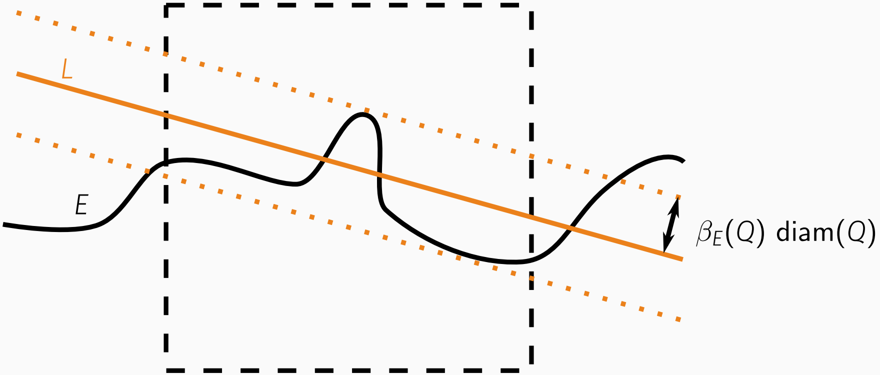


Quantitative rectifiability

β-numbers (Jones 1990)

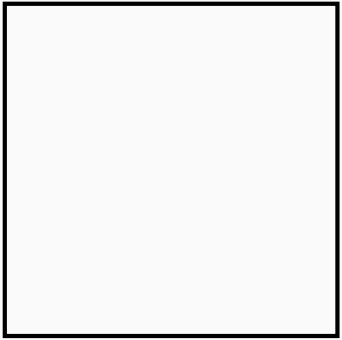
Given $E \subset \mathbb{R}^2$ and a square Q, $E \cap Q \neq \varnothing$, the **$\beta$ number** of E at Q is

$$\beta_E(Q) = \inf_L \sup_{x \in E \cap Q} \frac{\operatorname{dist}(x, L)}{\operatorname{diam}(Q)}.$$



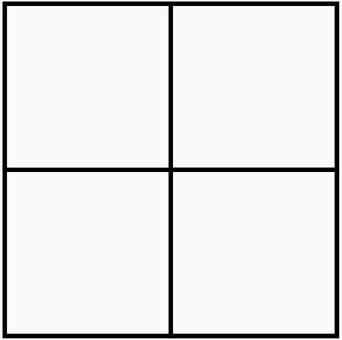
Analyst's traveling salesman theorem

dyadic lattice of squares $\mathcal{D}$ $\quad\leftrightsquigarrow\quad$ encoding scales and locations



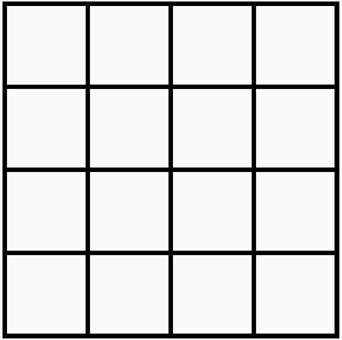
Analyst's traveling salesman theorem

dyadic lattice of squares $\mathcal{D}$ $\leftrightsquigarrow$ encoding scales and locations

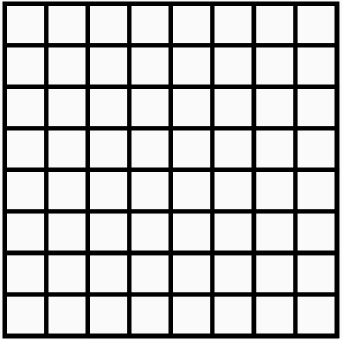


Analyst's traveling salesman theorem

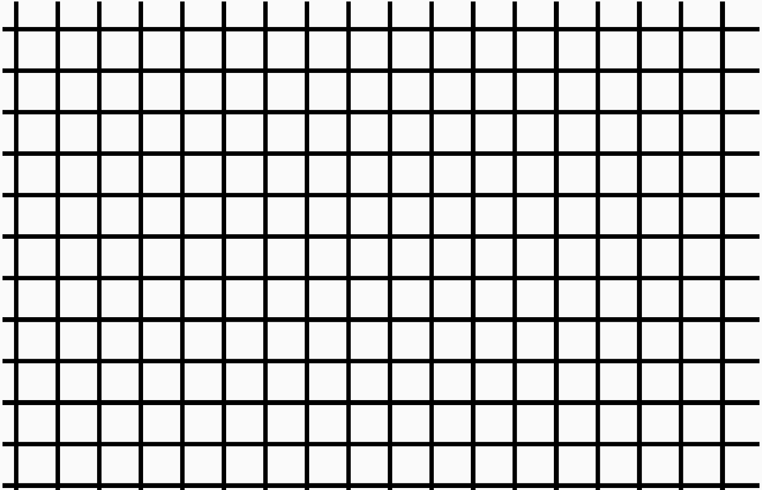
dyadic lattice of squares $\mathcal{D}$ $\leftrightsquigarrow$ encoding scales and locations



dyadic lattice of squares $\mathcal{D}$ $\leftrightsquigarrow$ encoding scales and locations



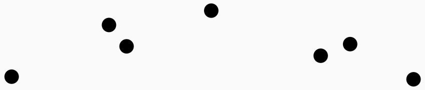
dyadic lattice of squares $\mathcal{D}$ $\quad\leftrightsquigarrow\quad$ encoding scales and locations



Analyst's traveling salesman theorem

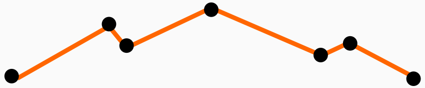
Analyst's traveling salesman theorem (Jones 1990)

A compact set $E \subset \mathbb{R}^2$ is contained in a rectifiable curve Γ if and only if



Analyst's traveling salesman theorem (Jones 1990)

A compact set $E \subset \mathbb{R}^2$ is contained in a rectifiable curve Γ if and only if



Analyst's traveling salesman theorem

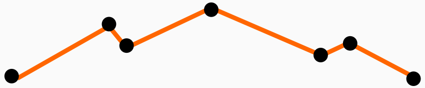
Analyst's traveling salesman theorem (Jones 1990)

A compact set $E \subset \mathbb{R}^2$ is contained in a rectifiable curve Γ if and only if

$$\sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty.$$

The length of the shortest such curve Γ satisfies

$$\operatorname{length}(\Gamma) \approx \operatorname{diam}(E) + \sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q).$$



Analyst's traveling salesman theorem

Analyst's traveling salesman theorem (Jones 1990)

A compact set $E \subset \mathbb{R}^2$ is contained in a rectifiable curve Γ if and only if

$$\sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty.$$

The length of the shortest such curve Γ satisfies

$$\operatorname{length}(\Gamma) \approx \operatorname{diam}(E) + \sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q).$$

$$\operatorname{length}(\Gamma_0) = \operatorname{diam}(E)$$

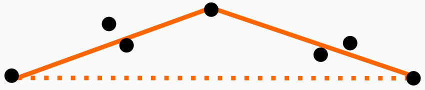
Analyst's traveling salesman theorem (Jones 1990)

A compact set $E \subset \mathbb{R}^2$ is contained in a rectifiable curve Γ if and only if

$$\sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty.$$

The length of the shortest such curve Γ satisfies

$$\operatorname{length}(\Gamma) \approx \operatorname{diam}(E) + \sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q).$$



$$\operatorname{length}(\Gamma_0) = \operatorname{diam}(E)$$

$$\operatorname{length}(\Gamma_1) - \operatorname{length}(\Gamma_0) \approx \beta_E(3Q)^2 \operatorname{diam}(Q)$$

Analyst's traveling salesman theorem

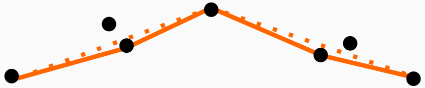
Analyst's traveling salesman theorem (Jones 1990)

A compact set $E \subset \mathbb{R}^2$ is contained in a rectifiable curve Γ if and only if

$$\sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty.$$

The length of the shortest such curve Γ satisfies

$$\operatorname{length}(\Gamma) \approx \operatorname{diam}(E) + \sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q).$$



$$\operatorname{length}(\Gamma_0) = \operatorname{diam}(E)$$

$$\operatorname{length}(\Gamma_1) - \operatorname{length}(\Gamma_0) \approx \beta_E(3Q)^2 \operatorname{diam}(Q)$$

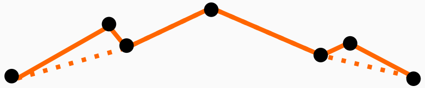
Analyst's traveling salesman theorem (Jones 1990)

A compact set $E \subset \mathbb{R}^2$ is contained in a rectifiable curve Γ if and only if

$$\sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q) < \infty.$$

The length of the shortest such curve Γ satisfies

$$\operatorname{length}(\Gamma) \approx \operatorname{diam}(E) + \sum_{Q \in \mathcal{D}} \beta_E(3Q)^2 \operatorname{diam}(Q).$$



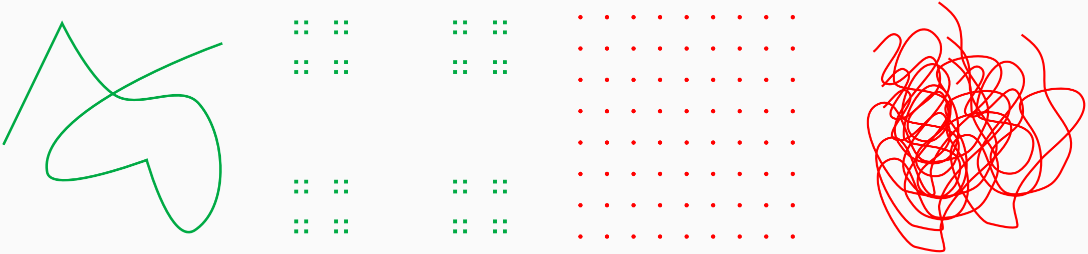
$$\operatorname{length}(\Gamma_0) = \operatorname{diam}(E)$$

$$\operatorname{length}(\Gamma_1) - \operatorname{length}(\Gamma_0) \approx \beta_E(3Q)^2 \operatorname{diam}(Q)$$

Uniformly rectifiable sets

We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for any $x \in E$, $0 < r < \text{diam}(E)$

$$cr \leq \text{length}(E \cap B(x,r)) \leq Cr.$$



Ahlfors regular non-Ahlfors regular

Uniformly rectifiable sets

We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for any $x \in E$, $0 < r < \mathrm{diam}(E)$

$$cr \leq \mathrm{length}(E \cap B(x,r)) \leq Cr.$$

Recall: a set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of rectifiable curves Γ_i such that

$$\mathrm{length}\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

Uniformly rectifiable sets

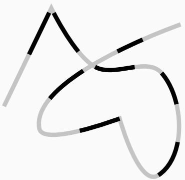
We say that $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for any $x \in E$, $0 < r < \text{diam}(E)$

$$cr \leq \text{length}(E \cap B(x,r)) \leq Cr.$$

Recall: a set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of rectifiable curves Γ_i such that

$$\text{length}\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

A set $E \subset \mathbb{R}^2$ is **uniformly rectifiable** if it is an Ahlfors regular subset of an Ahlfors regular curve.



Uniform rectifiability and SIOs

Theorem (David-Semmes 1991, 1993)

Let $E \subset \mathbb{R}^2$ be Ahlfors regular. The following are equivalent:

- E is uniformly rectifiable

Uniform rectifiability and SIOs

Theorem (David-Semmes 1991, 1993)

Let $E \subset \mathbb{R}^2$ be Ahlfors regular. The following are equivalent:

- E is uniformly rectifiable
- for every $R \in \mathcal{D}$

$$\sum_{Q \subset R} \beta_E(3Q)^2 \operatorname{diam}(Q) \leq C \operatorname{diam}(R)$$

Theorem (David-Semmes 1991, 1993)

Let $E \subset \mathbb{R}^2$ be Ahlfors regular. The following are equivalent:

- E is uniformly rectifiable
- for every $R \in \mathcal{D}$
$$\sum_{Q \subset R} \beta_E(3Q)^2 \operatorname{diam}(Q) \leq C \operatorname{diam}(R)$$
- E can be "well approximated by nice Lipschitz graphs"

Theorem (David-Semmes 1991, 1993)

Let $E \subset \mathbb{R}^2$ be Ahlfors regular. The following are equivalent:

- E is uniformly rectifiable
- for every $R \in \mathcal{D}$
$$\sum_{Q \subset R} \beta_E(3Q)^2 \operatorname{diam}(Q) \leq C \operatorname{diam}(R)$$
- E can be "well approximated by nice Lipschitz graphs"
- E defines many nice singular integral operators (including the Cauchy transform)

Solution of Vitushkin's conjecture

Vitushkin's conjecture (1967)

If E is purely unrectifiable and $\text{length}(E) < \infty$, then E is removable.

Solution of Vitushkin's conjecture

Vitushkin's conjecture (1967)

If E is purely unrectifiable and $\text{length}(E) < \infty$, then E is removable.

Theorem (Mattila-Melnikov-Verdera 1996)

If E is Ahflors regular and defines a bounded Cauchy transform, then it is uniformly rectifiable.

Consequently, Vitushkin's conjecture holds for Ahlfors regular sets.

Solution of Vitushkin's conjecture

Vitushkin's conjecture (1967)

If E is purely unrectifiable and $\mathrm{length}(E) < \infty$, then E is removable.

Theorem (Mattila-Melnikov-Verdera 1996)

If E is Ahflors regular and defines a bounded Cauchy transform, then it is uniformly rectifiable.

Consequently, Vitushkin's conjecture holds for Ahlfors regular sets.

Theorem (David 1998)

Vitushkin's conjecture holds for all sets with $\mathrm{length}(E) < \infty$.

Other applications of quantitative rectifiability

- solvability of elliptic equations with L^p-boundary data in domains with rough boundaries
  [Azzam, Hofmann, Mayboroda, Martell, Mourgoglou, Tolsa, Volberg]
- estimating size of singular sets for harmonic maps and other variational problems [Edelen-Naber-Valtorta]

My work in the area

Rectifiability (2017–2020)

Characterizations of rectifiability and uniform rectifiability using quantities similar to β-numbers.

My work in the area

Rectifiability (2017–2020)

Characterizations of rectifiability and uniform rectifiability using quantities similar to β-numbers.

Riesz transform (2019–2021, joint with Tolsa)

Geometric characterization of measures μ on $\mathbb{R}^n$ defining $L^2(\mu)$-bounded $(n-1)$-dimensional Riesz transforms

$\rightsquigarrow$ removable sets for Lipschitz harmonic functions in $\mathbb{R}^n$

My work in the area

Rectifiability (2017–2020)

Characterizations of rectifiability and uniform rectifiability using quantities similar to β-numbers.

Riesz transform (2019–2021, joint with Tolsa)

Geometric characterization of measures μ on $\mathbb{R}^n$ defining $L^2(\mu)$-bounded $(n-1)$-dimensional Riesz transforms

$\rightsquigarrow$ removable sets for Lipschitz harmonic functions in $\mathbb{R}^n$

Projections (2021–now, joint with Chang, Orponen, Villa)

Vitushkin's conjecture for sets of infinite length

Vitushkin's conjecture revisited

Vitushkin's conjecture

If E has $\mathbf{length}(E) < \infty$, then

$$E \text{ is removable} \quad \Leftrightarrow \quad E \text{ is purely unrectifiable.}$$

Vitushkin's conjecture revisited

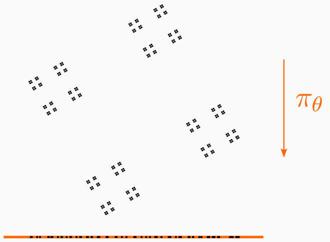
Vitushkin's conjecture

If E has $\text{length}(E) < \infty$, then

$$E \text{ is removable} \quad \Leftrightarrow \quad E \text{ is purely unrectifiable.}$$

Vitushkin's conjecture for general sets

$$E \text{ is removable} \quad \Leftrightarrow \quad \text{length}(\pi_\theta(E)) = 0 \text{ for a typical projection}$$



Vitushkin's conjecture revisited

Vitushkin's conjecture

If E has $\mathrm{length}(E) < \infty$, then

$$E \text{ is removable} \quad \Leftrightarrow \quad E \text{ is purely unrectifiable.}$$

Vitushkin's conjecture for general sets

$$E \text{ is removable} \quad \Leftrightarrow \quad \mathrm{length}(\pi_\theta(E)) = 0 \text{ for a typical projection}$$

The implication $\Leftarrow$ is false [Mattila '86, Jones-Murai '88], but the other implication is open.

Theorem (D. '24)

If E is Ahlfors regular and has big projections, then E is uniformly rectifiable.

This answered a question of David and Semmes from 1993.

Thank you!