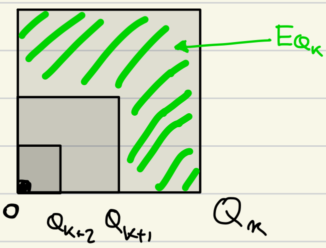
Singular Integral Operators

Exercise 5 - 28.11.2023

Exercise 1 (1 point). Prove that in $\mathbb{R}^n$ the family of all dyadic cubes containing 0 is $(1-2^{-n})$-sparse.

Solution. Let $\mathcal{S} := \{ Q \in \mathcal{D}(\mathbb{R}^n) : 0 \in Q \}$. Then, from the definition of dyadic cubes it follows that, if we define $Q_k := \prod_{j=1}^{n} [0, 2^{-k})$, then

$$\mathcal{S} = \{ Q_k : k \in \mathbb{Z} \}.$$



Hence, we define $E_{Q_k} := Q_k \setminus Q_{k+1}$, which is such that

$$|E_{Q_k}| = |Q_k| - |Q_{k+1}| = 2^{-nk} - 2^{-n(k+1)} = \left(1 - \frac{1}{2^n}\right) 2^{-nk}$$

$$= \left(1 - \frac{1}{2^n}\right) |Q_k|.$$

Moreover, $E_{Q_k} \cap E_{Q_j} = \emptyset$ for $k \neq j$, so $\mathcal{S}$ is a $\left(1 - \frac{1}{2^n}\right)$-sparse family.

$\square$

Exercise 2 (1 point)**.** Suppose that $\mathcal{F}_1, \dots, \mathcal{F}_k$ are sparse families of dyadic cubes, and that each $\mathcal{F}_j$ is η_j-sparse for some $\eta_j \in (0,1]$. Show that $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_k$ is $1/(\sum_{j=1}^k \eta_j^{-1})$-sparse.

Hint: Use Proposition 7.8.

Solution. By induction, it is enough to prove that the statement holds for two families $\mathcal{F}_1, \mathcal{F}_2$ of dyadic cubes, s.t. $\mathcal{F}_j$ is η_j-sparse $(j=1,2)$.

By Proposition 7.8 of the lecture notes, we have that $\mathcal{F}_j$ is $\frac{1}{\eta_j}$-Carleson $(j=1,2)$.

Thus, for every $R \in \mathcal{D}(\mathbb{R}^n)$ we have that

$$\sum_{Q \in \mathcal{F}_1 \cup \mathcal{F}_2 : Q \subset R} |Q| \leq \sum_{Q \in \mathcal{F}_1 : Q \subset R} |Q| + \sum_{Q \in \mathcal{F}_2 : Q \subset R} |Q|$$

$$\leq \left(\frac{1}{\eta_1} + \frac{1}{\eta_2} \right) |R| .$$

This shows that $\mathcal{F}_1 \cup \mathcal{F}_2$ is a $\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)$-Carleson family and, again by Proposition 7.8, in particular is a $\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^{-1}$-sparse family. $\square$

Exercise 3 (3 points)**.** For any $s \in (0,1)$ let $w_s = |x|^{1-s}$ be a weight on $\mathbb{R}$.

(i) Show that $w_s \in A_2$, and $[w_s]_{A_2} \leqslant s^{-1}$.

(ii) Given $f_s(x) = x^{s-1}\mathbf{1}_{(0,1)}(x)$, show that $\|f_s\|_{L^2(w_s)} \leqslant s^{-1/2}$.

(iii) Prove that $\|Hf_s\|_{L^2(w_s)} \geqslant Cs^{-3/2}$, and conclude that in the estimate from the A_2 theorem

$$\|Hf\|_{L^2(w)} \leqslant C[w]_{A_2}\|f\|_{L^2(w)},$$

the factor $[w]_{A_2}$ cannot be replaced by $[w]_{A_2}^t$ for any $t < 1$.

Solution.

(i) Let $s \in (0,1)$. We recall that

$$\boxed{[\omega]_{A_2} := \sup_{\substack{I \subset \mathbb{R} \\ \text{interval}}} \left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \omega^{-1}\right).}$$

We bound the supremum above by splitting the cases of I.

Case ① Assume that $I = [a,b]$, and that $b > a > 0$. Then

$$\frac{1}{|I|}\int_I \omega_s = \frac{1}{|b-a|}\int_a^b x^{1-s}\,dx = \frac{1}{2-s}\,\frac{b^{2-s}-a^{2-s}}{b-a}$$

and

$$\frac{1}{|I|}\int_I \omega_s^{-1} = \frac{1}{|b-a|}\int_a^b x^{s-1}\,dx = \frac{1}{s}\,\frac{(b^s-a^s)}{(b-a)},$$

so

$$\left(\frac{1}{|I|}\int_I \omega_s\right)\left(\frac{1}{|I|}\int_I \omega_s^{-1}\right) = \frac{1}{s(2-s)}\,\frac{(b^s-a^s)(b^{2-s}-a^{2-s})}{(b-a)^2} \overset{\text{mean. val. thm.}}{\leq} \frac{1}{s(2-s)}\,\frac{(b^s-a^s)(2-s)\,b^{1-s}(b-a)}{(b-a)^2}$$

$$= \frac{1}{s}\,\frac{(b-a^s\underbrace{b^{1-s}}_{\geq a^{1-s}})}{(b-a)} \leq \frac{1}{s}\,\frac{(b-a)}{(b-a)} = \frac{1}{s}.$$

Case ② Assume that $b > a > 0$, and that $I = [-a,b]$. Then

$$\int_I \omega_s = \int_0^a \omega_s + \int_0^b \omega_s = \frac{1}{(2-s)}\left(b^{2-s}+a^{2-s}\right).$$

Moreover

$$\int_I \omega_s^{-1} = \int_0^a \omega_s^{-1} + \int_0^b \omega_s^{-1} = \frac{1}{s}\left(b^s+a^s\right).$$

So, observing that $|I| = (b+a)$, we have

$$\left(\frac{1}{|I|}\int_I \omega_s\right)\left(\frac{1}{|I|}\int_I \omega_s^{-1}\right) = \frac{1}{(b+a)^2}\,\frac{(b^{2-s}+a^{2-s})}{(2-s)}\,\frac{(b^s+a^s)}{s}$$

$$\leq \frac{1}{(b+a)^2}\,\frac{\overset{(1)}{(b+a)^{2-s}}}{(2-s)}\,\frac{\overset{(2)}{2^s(b+a)^s}}{s} = \underbrace{\frac{2^s}{(2-s)}}_{\leq 1}\,\frac{1}{s} \leq \frac{1}{s},$$

where (1) and (2) follows by Minkowski and Hölder's inequality (for sums) respectively.

Finally, we observe that $\omega_s(\cdot)$ is an <u>even</u> function, so Cases (1) and (2) allow us to conclude the proof of (i).

(ii) A standard computation yields that

$$\|f_s\|^2_{L^2(\omega_s)} = \int_0^1 (x^{s-1})^2\omega_s(x)\,dx = \int_0^1 x^{s-1}\,dx = \frac{1}{s},$$

so $\|f_s\|_{L^2(\omega_s)} = s^{1/2}$, which proves point (ii).

(iii) Let us first claim that

$$\|Hf_s\|_{L^2(\omega_s)} \geq \tilde{c}\, s^{-3/2} \qquad (*)$$

holds. This, together with (i)-(ii) readily implies that, if

$$\|Hf_s\|_{L^2(\omega_s)} \leq C_\alpha [\omega_s]^{\alpha}_{A_2} \|f_s\|_{L^2(\omega_s)}$$

holds, then

$$\tilde{c}\, s^{-3/2} \leq C_\alpha\, s^{-1-\frac{\alpha}{2}} \qquad \tilde{c}\, C_\alpha^{-1} \leq s^{\frac{1-\alpha}{2}},$$

which gives a contradiction for $s \to 0$ if $\alpha \neq 1$.

We are left with proving the claim (*). We first observe that, if $x \geq 2$ and $y \in (0,1)$, then trivially $0 < x-y \leq x$. Then:

$$\|Hf_s\|^2_{L^2(\omega_s)} \geq \int_2^{+\infty} |Hf_s(x)|^2 \omega_s(x)\, dx = \frac{1}{\pi^2}\int_2^{\infty} \left| \int_0^1 \frac{y^{s-1}}{x-y}\, dy \right|^2 |x|^{1-s}\, dx$$

$$\geq \frac{1}{\pi^2}\int_2^{\infty} \left| \int_0^1 \frac{y^{s-1}}{x}\, dy \right|^2 x^{1-s}\, dx$$

$$= \frac{1}{\pi^2}\left(\int_2^{\infty} \frac{1}{x^{1+s}}\, dx\right)\left(\int_0^1 y^{s-1} dy\right)^2 = \frac{1}{\pi^2}\,\frac{1}{s}\,\frac{1}{2^s}\,\frac{1}{s^2} \geq \frac{1}{(2\pi)^2}\,\frac{1}{s^3},$$

where we used the fact that, by Prop. 5.6 in the lecture notes, the fact that $f_s \in L^1(\mathbb{R})$ implies that $H_\varepsilon f \xrightarrow[\varepsilon \to 0]{} Hf$ pointwise $\mathcal{L}^1$-a.e.

Thus, $\|Hf_s\|_{L^2(\omega_s)} \geq (2\pi)^{-1} s^{-3/2}$, which proves the claim (*). $\square$

Remark: For Ex. 3 we also refer to p. 1374 of the article

S. Petermichl, "The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristics", Am. J. Math. 129, No 5, 1355 – 1375 (2007).