Singular Integral Operators

Exercise 4 - 21.11.2023

Exercise 1 (1 point). Let $1 \leqslant p < \infty$ and $w \in A_p$. Show that $L^1(\mathbb{R}^n)$ is dense in $L^p(w)$.
Hint: For any $f \in L^p(w)$ prove that $f_R := f\mathbf{1}_{B(0,R)} \in L^1(\mathbb{R}^n)$ for all $R > 0$, and that $f_R \to f$ in $L^p(w)$ as $R \to \infty$. The estimate (6.9) from lecture notes (and its modification for $p = 1$) may be helpful.

Solution. We study the cases $p=1$ and $p>1$ separately. Also, we denote as $Q_R := Q(0,R)$ the cube centered at 0, with side-length $R>0$, and whose sides are parallel to the coordinate axis.

We follow the hint, and first prove that $f_R := f\, \mathbb{1}_{Q(0,R)} \in L^1(\mathbb{R}^n)$ $\forall\ R>0$.

Case $p=1$. The (A_1) condition readily implies that

$$\frac{\omega(Q_R)}{|Q_R|}|f(x)| \lesssim |f(x)|\,\omega(x) \quad \text{for a.e. } x \in Q_R.$$

Hence, we have that $\displaystyle\int f\, \chi_{Q_R} \frac{\omega}{\omega} \lesssim \frac{|Q_R|}{\omega(Q_R)} \int_{Q_R} f\,\omega < \infty \Rightarrow f_R \in L^1(\omega)\ \forall R>0.$

Case $1<p<\infty$. Let $R>0$. An application of Hölder's inequality together with the (A_p) condition gives that, for $f \in L^p(\omega)$, it holds:

$$\int |f|\chi_{B_R} = \int |f|\,\omega^{\frac{1}{p}}\,\omega^{-\frac{1}{p}}\chi_{Q_R} \underset{\text{Hölder}}{\leq} \left(\int |f|^p \omega\right)^{\frac{1}{p}} \left(\int \chi_{Q_R}\,\omega^{1-p'}\right)^{\frac{p-1}{p}}$$

$$\leq \|f\|_{L^p(\omega)} \left(\frac{1}{|Q_R|}\int_{Q_R} \omega^{1-p'}\right)^{\frac{p-1}{p}} |Q_R|^{\frac{p-1}{p}} \underset{(A_1)}{\lesssim} \|f\|_{L^p(\omega)} \frac{|Q_R|^{\frac{1}{p}}}{\omega(Q_R)} \frac{|Q_R|^{1-\frac{1}{p}}}{\omega(Q_R)^{\frac{1}{p}}}$$

$$= \|f\|_{L^p(\omega)} \frac{|Q_R|}{\omega(Q_R)^{\frac{1}{p'}}} < \infty$$

Hence, $f_R = f \chi_{B_R}$ belongs to $L^1(\mathbb{R}^n)$ $\forall R>0$.

We are left with proving that $f_R \to f$ in $L^p(\omega)$ as $R\to\infty$, for $1\le p<\infty$. This follows from the fact that

$$\int |f_R - f|^p \,\omega = \int_{\mathbb{R}^n \setminus Q_R} |f|^p \,\omega , \qquad (*)$$

where the right-hand side of $(*)$ tends to 0 as $R\to+\infty$ because $f\in L^p(\omega)$. This concludes the exercise. $\square$

Exercise 2 (1 point)**.** Prove that in the definition of the A_p condition we may replace cubes by balls and still get the same class of weights. More specifically,

$$S_1 := \sup_{Q\subset\mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} \sim \sup_{B\subset\mathbb{R}^n} \left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B w^{1-p'}\right)^{p-1} =: S_2$$

where Q are cubes and B are balls.

Hint: The doubling condition ($w(2B) \leqslant Cw(B)$ for all balls) is relevant.

C_{doub}

Solution. For a cube $Q\subset\mathbb{R}^n$, we denote as x_Q and $\ell(Q)$ its center and side-length, respectively. Then, we observe that there exists $N = N(n)\in\mathbb{N}$ such that, for $c=2^N$ it holds:

$$B_Q \subset Q \subset cB_Q \quad \forall \text{ cube } Q\subset\mathbb{R}^n. \quad (*)$$

In particular, by the doubling condition of ω, we have

$$\omega(Q) \leq \omega(cB_Q) \leq C_{doub}^N\, \omega(B_Q) \leq C_{doub}^N\, \omega(Q). \quad (**)$$

Hence, for $p\in(1,\infty)$, it holds that

$$\left(\frac{1}{|Q|}\int_Q \omega^{1-p'}\right)^{p-1} = \left(\frac{1}{|Q|}\int_Q \omega^{-\frac{p'}{p}}\right)^{p-1} \lesssim \frac{1}{|cB_Q|^{p-1}}\left(\int_{cB_Q} \omega^{-\frac{p'}{p}}\right)^{p-1} \quad (***)$$

Thus

$$\left(\frac{1}{|Q|}\int_Q \omega^{1-p'}\right)^{p-1} \lesssim S_2 \frac{|cB_Q|}{\omega(cB_Q)} \sim S_2 \frac{|Q|}{\omega(Q)},$$

where the last bound follows from $(**)$. Hence

$$\left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{1-p'}\right)^{p-1} \lesssim S_2 \Rightarrow \boxed{S_1 \lesssim S_2.}$$

The proof of $\boxed{\text{"}S_2 \lesssim S_1\text{"}}$ is completely analogous. $\square$

Exercise 3 (2 points)**.** Prove that $w(x) = |x|^a$ is an A_p weight on $\mathbb{R}^n$, $1 < p < \infty$, if and only if $-n < a < n(p-1)$.

Solution. We first prove that, if $a \in (-n, n(p-1))$, then w is an A_p weight.

Before presenting the proof, we remark that $|x|^a$ is radially increasing (resp. decreasing) for $a \geq 0$ (resp. for $a < 0$).

Recall that the (A_p) condition reads

$$(*) \qquad \sup_{B(x_0,r) \text{ balls}} \left(\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |x|^a \, dx \right) \left(\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |x|^{-a\frac{p'}{p}} \, dx \right)^{\frac{p}{p'}} < \infty$$

We study the quantity in $(*)$ dividing in the cases.

Case 1: $|x_0| \geq 3r$. For $y \in B(x_0, r)$, it holds

$$\boxed{(**)} \qquad \boxed{\frac{2}{3}|x_0| \leq |x_0| - r \leq |y| \leq |x_0| + r \leq 2|x_0| .}$$

Hence, we have

$$\int_{B(x_0,r)} |x|^a \, dx \sim \int_{B(x_0,r)} |x_0|^a \, dx \sim |B(x_0,r)| \, |x_0|^a$$

and

$$\int_{B(x_0,r)} |x|^{-a\frac{p'}{p}} \, dx \sim \int_{B(x_0,r)} |x_0|^{-a\frac{p'}{p}} \, dx \sim |B(x_0,r)| \, |x_0|^{-a\frac{p'}{p}}.$$

Hence

$$\left(\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |x|^a dx \right) \left(\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |x|^{-a\frac{p'}{p}} dx \right)^{\frac{p}{p'}} \approx |x_0|^a \, |x_0|^{-a} = 1$$

This shows that the claim $(*)$ holds for balls s.t. $|x_0| \geq 3r$.

Case 2: $|x_0| < 3r$. In this case we have $B(x_0, r) \subseteq B(0,5r)$.

We further observe that $|B(x_0,r)| \sim |B(0,5r)|$.

Thus

$$\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |x|^a dx\right)\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |x|^{-a\frac{p'}{p}} dx\right)^{\frac{p}{p'}} \lesssim \left(\frac{1}{|B(0,5r)|}\int_{B(0,5r)} |x|^a dx\right)\left(\frac{1}{|B(0,5r)|}\int_{B(0,5r)} |x|^{-a\frac{p'}{p}} dx\right)^{\frac{p}{p'}} =: (***)$$

One can easily see that $(***) < \infty$ iff $a > -n$ and $-a\frac{p'}{p} < n$, namely iff $-n < a < n\frac{p}{p'}$. In this case, we have

$$(***) \sim \left(\frac{1}{r^n} r^{n+a}\right)\left(\frac{1}{r^n} r^{n-a\frac{p'}{p}}\right)^{\frac{p}{p'}} \sim r^a r^{-a} \sim 1 .$$

Hence, we conclude the proof that $-n < a < n\frac{p}{p'} \Rightarrow \omega \in A_p$.

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Conversely, if ω is an A_p weight, then the term $(*)$ must be finite which, as we showed, implies that $a \in \left(-n, n\frac{p}{p'}\right)$. $\square$

Exercise 4 (1 point)**.** Show that

$$w(x) = \begin{cases} \log \frac{1}{|x|} & |x| \leqslant e^{-1} \\ 1 & |x| > e^{-1} \end{cases}$$

is an A_1 weight on $\mathbb{R}^n$.

Solution. We first observe that, in order to prove the A_1-conditions, it is enough to check that $M\omega(x_0) \lesssim \omega(x_0)$ for a.e. $x_0 \in \mathbb{R}^n$ or, equivalently, that for a.e. $x_0 \in \mathbb{R}^n$ we have

$$\omega(B(x_0,r)) \lesssim \omega(x_0)\,|B(x_0,r)| \quad \forall\, r>0.$$

We study cases separately as in the previous exercise.

Case 1: $|x_0| \geq 3r$. Triangle inequality implies that, if $x \in B(x_0,r)$, it holds

$$\frac{2}{3}|x_0| \leq |x_0| - r \leq |x| \leq |x_0| + r \leq 2|x_0|. \qquad (*)$$

Now, we write

$$\omega(B(x_0,r)) = \int_{B(x_0,r)\cap B(0,e^{-1})} \log \frac{1}{|x|}\,dx + |B(x_0,r) \setminus B(0,e^{-1})|.$$

(1.1) For $|x_0|<e^{-1}$, $(*)$ yields

$$\boxed{\log\left(\frac{1}{|x|}\right)} \leq \log\left(\frac{3}{|x_0|}\right) \lesssim \boxed{\log\left(\frac{1}{|x_0|}\right)} \lesssim \log\left(\frac{2}{|x|}\right) \lesssim \boxed{\log\left(\frac{1}{|x|}\right)}.$$

Hence

$$\omega(B(x_0,r)) \lesssim \log\frac{1}{|x_0|}\,|B(x_0,r)\cap B(0,e^{-1})| + |B(x_0,r)\setminus B(0,e^{-1})|$$
$$\lesssim \log\frac{1}{|x_0|}\,|B(x_0,r)| = \omega(x_0)\,|B(x_0,r)|.$$

(1.2) For $|x_0| \geq e^{-1}$ we have that, for $x \in B(x_0,r)$, $(*)$ implies that

$$\log\left(\frac{1}{|x|}\right) \leq \log\left(\frac{3}{2|x_0|}\right) \leq \log\left(\frac{3}{2}e\right),$$

So

$$\omega(B(x_0,r)) \lesssim \log\left(\frac{3}{2}e\right)|B(x_0,r)\cap B(0,e^{-1})| + |B(x_0,r)\setminus B(0,e^{-1})|$$
$$\lesssim |B(x_0,r)| = \omega(x_0)\,|B(x_0,r)|.$$

Case 2: $|x_0| < 3r$. In this case we have that $B(x_0, r) \subseteq B(0, 5r)$,

and
$$\left| \log \frac{1}{r} \right| \lesssim \left| \log\left(\frac{3}{|x_0|}\right) \right| .$$

Moreover
$$\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} \omega(x)\, dx \lesssim \frac{1}{|B(0,5r)|} \int_{B(0,5r)} \omega(x)\, dx =: (*) ,$$

and, for $5r < e^{-1}$:
$$\int_{B(0,5r)} \log \frac{1}{|x|}\, dx \lesssim \int_0^{5r} \log \frac{1}{s}\, s^{n-1} ds \lesssim r^{n-1} \int_0^{5r} \log \frac{1}{s}\, ds \lesssim \left(\log \frac{1}{r}\right) r^n ,$$

where in the last inequality we used the fact that, for $r \ll 1$, it holds:
$$\left| \int_0^r \log\left(\frac{1}{s}\right) ds \right| \overset{t=\frac{1}{s}}{=} \left| \int_{\frac{1}{r}}^{\infty} \frac{\log t}{t^2}\, dt \right| \leq \frac{\log\left(\frac{1}{r}\right)}{\left(\frac{1}{r}\right)^{1/2}} \int_{\frac{1}{r}}^{\infty} \frac{1}{t^{3/2}}\, dt \sim r \log \frac{1}{r} .$$

On the other hand, if $5r > e^{-1}$, then
$$\int_{B(0,5r)} \omega(x) = \int_{B(0,e^{-1})} \log \frac{1}{|x|}\, dx + |B(0,5r) \setminus B(0,e^{-1})| \lesssim r^n .$$

Hence:
$$\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} \omega(x)\, dx \lesssim \begin{cases} \log\left(\frac{1}{r}\right) \leq \log\left(\frac{1}{3|x_0|}\right) \sim \omega(x_0) , & \text{if } 5r < e^{-1} \\ 1 \sim \omega(x_0) , & \text{if } 5r \geq e^{-1} . \end{cases}$$

In conclusion, we have proved that
$$M\omega(x_0) \lesssim \omega(x_0) \qquad \text{for a.e. } x_0 \in \mathbb{R}^n$$

which, recalling that $M_c\,\omega(x_0) \lesssim M\omega(x_0)$, implies that the (A_1) condition holds by [Exercise 4, Ex. Sheet 3] $\square$