Singular Integral Operators

Exercise 3 - 14.11.2023

Exercise 1 (1 point). Prove that if a Calderón-Zygmund operator T is associated to a standard kernel K, then its adjoint (see Section 4.3 in the notes) is also a Calderón-Zygmund operator, and it is associated to the standard kernel

$$K^t(x,y) = \overline{K(y,x)}.$$

→ See [Parissis, Lemma 6.13]

Solution. We first recall that T^t is such that

$$\int T^t(g)\,\overline{f} = \int g\,\overline{T(f)} \qquad \forall\ f,g \in L^2(\mathbb{R}^n).$$

We check that T^t satisfies all the properties of CZO (Definition 4.9 of the lecture notes).

(i) T^t is of strong type $(2,2)$: this is an immediate consequence of T being a CZO and, in particular, of strong type $(2,2)$.

(ii) $K^t(x,y) := \overline{K(y,x)}$ is a standard kernel: it readily follows from $K(x,y)$ being a standard kernel.

(iii) T^t is associated to K^t: We first observe that if $f,g \in L^2(\mathbb{R}^n)$ are such that $\boxed{\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset}$, then

$$\langle Tf, g\rangle_{L^2} = \int Tf(x)\,\overline{g(x)}\,dx = \iint K(x,y)\,f(y)\,\overline{g(x)}\,dy\,dx \qquad (*)$$
$$= \int f(y)\left(\int \overline{K(x,y)\,g(x)}\,dx\right) dy,$$

where in the second equality we used that T is associated to K and the last equality follows from Fubini's theorem.

In the case of a general function $g \in L^2(\mathbb{R}^n)$ with compact support, assume that $z \in \mathbb{R}^n \setminus \mathrm{supp}(g)$, and consider $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\mathrm{supp}(\phi) \subset B(0,1)$ and $\int \phi = 1$. Then, for $\varepsilon > 0$ we define

$$\phi_\varepsilon(y) := \phi\left(\frac{y-z}{\varepsilon}\right),$$

so that, for ε small enough, $\mathrm{supp}(\phi_\varepsilon) \subset B(z,\varepsilon) \subset \mathbb{R}^n \setminus \mathrm{supp}(g)$.

Then, we apply $(*)$ with $f(y) = \phi_\varepsilon(y)$, and obtain

$$\int \phi_\varepsilon(z-y)\, \overline{T^t(g)(y)}\, dy = \int \phi_\varepsilon(z-y) \overline{\int \overline{K(x,y)}\, g(x)\, dx}\, dy$$

which, for $\varepsilon \to 0$, yields

$$T^t(g)(z) = \int \overline{K(x,z)}\, g(x)\, dx \quad \text{for a.e. } z \in \mathrm{supp}(g),$$

which concludes the proof. $\square$

The following exercise demonstrates that the strong type $(2,2)$ estimate in the definition of Calderón-Zygmund operators can be replaced by any other strong type (p,p) estimate, $1 < p < \infty$, and the class of operators remains the same.

Exercise 2 (1 point). Suppose that in the definition of Calderón-Zygmund operators (Definition 4.9) we replaced the strong type $(2,2)$ estimate by the strong type (p,p) estimate for some $1 < p < \infty$. How would you modify the proof of Theorem 4.13 to get that these "new" Calderón-Zygmund operators are weak type $(1,1)$ and strong type (q,q), $1 < q < \infty$?[1] You don't have to repeat the whole proof, just the parts that change compared to $p = 2$.

Solution. As suggested in the exercise, we will only focus on the differences in the proof.

1- Proof of the weak type (1,1) estimate. Let $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and let $\alpha > 0$. We apply again Calderón-Zygmund decomposition to f at level α, which gives $f = g + b$. Hence, $Tf = Tg + Tb$, and

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \alpha\}| \le |\{x \in \mathbb{R}^n : |Tg(x)| > \tfrac{\alpha}{2}\}| + |\{x \in \mathbb{R}^n : |Tb(x)| > \tfrac{\alpha}{2}\}| =: I_g + I_b .$$

The bounds $\|g\|_{L^1} \le \|f\|_{L^1}$ and $\|g\|_{L^\infty} \le 2^n \alpha$ together with (the L^p-version of) Chebyshev's inequality and the strong type (p,p)-boundedness imply:

$$I_g \le \frac{\|Tg\|^p_{L^p(\mathbb{R}^n)}}{(\frac{\alpha}{2})^p} \lesssim \frac{\|g\|^p_{L^p(\mathbb{R}^n)}}{\alpha^p} \le \frac{\|g\|^{p-1}_{L^\infty(\mathbb{R}^n)}}{\alpha^{p-1}} \frac{\|g\|_{L^1(\mathbb{R}^n)}}{\alpha}$$

$$\le 2^{(p-1)n} \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\alpha} .$$

In order to bound I_b, it is enough to repeat the proof in the lecture notes, as it does not use the strong-type boundedness. In particular, we have:

$$I_b \lesssim \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\alpha} .$$

2- Weak type (1,1) implies strong type (q,q). Similar argument (by duality) via Marcinkiewicz interpolation theorem and duality as in the lecture notes. $\square$

singular integral

Exercise 3 (2 points). Let T be a Calderón-Zygmund operator, and T_ε the associated truncated operators. Prove that for $f \in L^1(\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0} T_\varepsilon f(x) = Tf(x) \qquad \text{for a.e. } x \in \mathbb{R}^n, \tag{0.1}$$

and for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$\lim_{\varepsilon\to 0} \|T_\varepsilon f - Tf\|_{L^p} = 0. \tag{0.2}$$

Hint: For (0.1) use the weak type $(1,1)$ estimates of T and the maximal operator T_*. For (0.2) use that $\lim_{\varepsilon\to 0} T_\varepsilon f(x) = Tf(x)$ for a.e. $x \in \mathbb{R}^n$ (shown in the lecture) and the dominated convergence theorem.

Solution.

Proof of (0.1). We follow the proof of Proposition 5.6 in the lecture notes. For $f \in L^1(\mathbb{R}^n)$ and $\varepsilon > 0$, we define

$$\Lambda_\varepsilon f(x) := |T_\varepsilon f(x) - Tf(x)| \quad \text{and} \quad \Lambda f(x) := \limsup_{\varepsilon\to 0} \Lambda_\varepsilon f(x).$$

As in the notes, we assume that f_n is a sequence in $L^1(\mathbb{R}^n)$ s.t. $f_n \to f$ in $L^1(\mathbb{R}^n)$.

In particular, we have

$$\Lambda f(x) \le \Lambda(f - f_n)(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Moreover, $\Lambda g \le T_* g + Tg$ for all $g \in L^1(\mathbb{R}^n)$. Hence, for $\lambda > 0$ we have

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n : \lim_{\varepsilon\to 0} T_\varepsilon f(x) \neq Tf(x)\right\}\right| \\
&\le |\{x \in \mathbb{R}^n : \Lambda f > \lambda\}| \le |\{x \in \mathbb{R}^n : \Lambda(f - f_n) > \lambda\}| \\
&\le \left|\left\{x \in \mathbb{R}^n : T_*(f - f_n) > \tfrac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : |T(f - f_n)| > \tfrac{\lambda}{2}\right\}\right| \\
&\lesssim \frac{\|f - f_n\|_{L^1(\mathbb{R}^n)}}{\lambda}, \qquad (*)
\end{aligned}$$

where the last bound follows from the weak type $(1,1)$ bounds of T_* and T.

We take the limit in $(*)$ for $n \to \infty$, and we obtain (0.1).

<u>Proof of **(0.2)**</u>. Let $f \in L^p(\mathbb{R}^n)$, $1<p<\infty$. The operators T_* and T are bounded on $L^p(\mathbb{R}^n)$, which implies that for every $\varepsilon>0$ we have

$$\int |T_\varepsilon f - Tf|^p \leq \int (|T_\varepsilon f| + |Tf|)^p \leq \int (|T_* f| + |Tf|)^p \lesssim \int |f|^p.$$

Hence, dominated convergence theorem yields that

$$\lim_{\varepsilon \to 0} \| T_\varepsilon f - Tf \|_{L^p(\mathbb{R}^n)} = \| \lim_{\varepsilon \to 0} (T_\varepsilon f - Tf) \|_{L^p(\mathbb{R}^n)} = 0,$$

where the last equality follows from the fact that $\lim_{\varepsilon \to 0} T_\varepsilon f(x) = Tf(x)$ for every $f \in L^p(\mathbb{R}^n)$ and a.e. $x \in \mathbb{R}^n$. $\square$

Exercise 4 (1 point). Show that there exists $C = C(n) \geqslant 1$ such that for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have

$$C^{-1}Mf(x) \leqslant M_c f(x) \leqslant CMf(x). \qquad (*)$$

Conclude that M is of weak type (p,p) with respect to a weight w for some $1 \leqslant p < \infty$ if and only if M_c is of weak type (p,p) with respect to w.

Solution. We first prove $(*)$. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

For every cube $Q \subset \mathbb{R}^n$ such that $x \in Q$, it holds

$$\frac{1}{|Q|}\int_Q |f| \leq \frac{1}{|Q|}\int_{B(x,\, diam(Q))} |f| \sim_n \frac{1}{|B(x, diam(Q))|}\int_{B(x, diam(Q))} |f|$$

def. of Mf

$$\leq Mf(x).$$

Hence, taking the supremum over $Q \ni x$ we obtain $\boxed{M_c f(x) \lesssim_n Mf(x)}$.

Conversely, we observe that $\exists\, \tilde{C} = \tilde{C}(n) > 1$ such that for every ball $B(x,r) \subset \mathbb{R}^n$ we can find a cube $Q_{x,r}$ centered at x and such that

$$B(x,r) \subseteq Q_{x,r} \subseteq B(x, \tilde{C} r).$$

In particular, we have $|Q_{x,r}| \sim_n |B(x,r)|$ and

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy \lesssim_n \frac{1}{|Q_{x,r}|}\int_{Q_{x,r}} |f(y)|\,dy \leq M_c f(x). \qquad (**)$$

We take the sup. over $r > 0$ in $(**)$, and conclude the proof of $(*)$.

As far as the second part of the statement is concerned, it is enough to observe that the weight w is <u>non-negative</u> so $(*)$ implies that, for $f \in L^p(\mathbb{R}^n)$, it holds:

$$w(\{x \in \mathbb{R}^n : M_c f(x) > \lambda\}) \leq w(\{x \in \mathbb{R}^n : Mf(x) > C^{-1}\lambda\})$$

and

$$w(\{x \in \mathbb{R}^n : M f(x) > \lambda\}) \leq w(\{x \in \mathbb{R}^n : M_c f(x) > C^{-1}\lambda\}).$$

The bounds above conclude the proof (see Def. 6.1 of the lecture notes). $\square$

Exercise 5 (1 point). Show that the A_1 condition is equivalent to

$$M_c w(x) \leqslant C w(x) \quad \text{for a.e. } x \in \mathbb{R}^n. \qquad \boxed{(A_1^*)}$$

Solution. We recall that ω is an A_1-weight if $\exists\ \tilde{C} > 0$ s.t. for every cube $Q \subset \mathbb{R}^n$ it holds

$$\frac{\omega(Q)}{|Q|} \leq \tilde{C}\, \omega(x) \quad \text{for a.e. } x \in Q. \qquad \boxed{(A_1)}$$

"$(A_1^*) \Rightarrow (A_1)$" We first observe that, if $x \in \mathbb{R}^n$ is such that (A_1^*) holds and $Q \subset \mathbb{R}^n$ is a cube containing x, then (A_1^*) gives

$$\frac{\omega(Q)}{|Q|} \leq M_c \omega(x) \leq \tilde{C}\, \omega(x).$$

"$(A_1) \Rightarrow (A_1^*)$" Assume that (A_1) holds. Then, we define $A_0 := \{x \in \mathbb{R}^n : M_c\omega(x) > c\,\omega(x)\}$ and we claim that $|A_0| = 0$.

Let $x \in \mathbb{R}^n$ be a point such that

$$M_c\, \omega(x) > c\, \omega(x). \qquad (*)$$

Hence, there exists a cube Q s.t. $x \in Q$ and

$$\frac{\omega(Q)}{|Q|} > c\,\omega(x). \qquad (**)$$

Furthermore

$$c\,\omega(x) \overset{(**)}{<} \frac{\omega(Q)}{|Q|} \overset{(A_1)}{<} c \operatorname{ess-inf} \omega(x),$$

which implies that $\omega(x) < \operatorname*{ess-inf}_{y \in Q} \omega(y)$. In particular, the set $E_Q := \{x \in Q : (**) \text{ holds}\}$ is Lebesgue-null. Furthermore, for every cube $Q \subset \mathbb{R}^n$ and $0 < \varepsilon \ll 1$, there exists a cube $\tilde{Q} \subset \mathbb{R}^n$ s.t. $Q \subset \tilde{Q}$, its corners have rational coordinates, and $|\tilde{Q}| < |Q| + \varepsilon$.

Furthermore, for ε small enough, it holds

$$\frac{\omega(\tilde{Q})}{|\tilde{Q}|} \geq \frac{1}{|Q|+\varepsilon}\, \omega(Q) \overset{(**)}{>} c\,\omega(x).$$

The cubes with rational corners are <u>countable</u>, so we can cover A_θ by a countable union of cubes Q_i with the property $(*)$:

$$A_\theta \subseteq \bigcup_{i=1}^{\infty} (Q_i \cap E_{Q_i}).$$

Hence

$$|A_\theta| \leq \sum_{i=1}^{\infty} |Q \cap E_{Q_i}| = 0,$$

which concludes the proof. $\square$