Singular Integral Operators

Exercise 1 - 31.10.2023

Exercise 1 (1 point). Let $f \in \mathcal{S}(\mathbb{R})$. Show that $Hf \in L^1(\mathbb{R})$ if and only if $\int_{\mathbb{R}} f(y)\, dy = 0$.
Hint: Modify the proof of Lemma 3.5 from the lecture notes to estimate the asymptotics of $x^2 \cdot Hf(x)$ as $|x| \to \infty$.

Solution. Let $f \in \mathcal{S}(\mathbb{R})$, $x \in \mathbb{R}$ such that $|x| > 100$, and $0 < \varepsilon \ll 1$.

$\boxed{\text{Aim}}$: We want to prove that

$$\boxed{\lim_{|x| \to \infty} \left| x^2 H_\varepsilon f(x) - \frac{x}{\pi} \int_{\mathbb{R}} f(x-y)\, dy \right| \to 0, \quad \varepsilon \to 0,} \qquad (*)$$

which shows that
$$\lim_{|x| \to \infty} \left| x^2 Hf(x) - \frac{x}{\pi} \int_{\mathbb{R}} f(x-y)\, dy \right| = 0. \qquad (**)$$

Before proving $\boxed{*}$, we observe that $\boxed{**}$ concludes the exercise: it is enough to argue as in Corollary 3.6 of the lecture notes.

Hence, we turn to the proof of $(*)$. First, we write

$$\pi\, x^2 \cdot H_\varepsilon f(x) = x^2 \int_{|y| > \varepsilon} \frac{f(x-y)}{y}\, dy = \int_{\varepsilon < |y| < \frac{|x|}{2}} \frac{x^2 \left[f(x-y) - f(x) \right]}{y}\, dy$$
$$+ x^2 \int_{\frac{|x|}{2} \le |y| < 2|x|} \frac{f(x-y)}{y}\, dy + \int_{|y| \ge 2|x|} \frac{f(x-y)}{y}\, dy$$
$$=: I_{1,\varepsilon} + I_2 + I_3$$

Hence, we estimate the terms separately.

$I_{1,\varepsilon}$ First observe that, for $\xi \in B(x,y)$ we have:

$$\xi \in B(x,y) \Rightarrow |\xi| \le |x| + |y| \le \frac{3}{2}|x|,$$
$$|\xi| \ge |x| - |y| > \frac{|x|}{2}.$$

Thus:

$$I_{1,\varepsilon} \leq \int_{\varepsilon<|y|<\frac{|x|}{2}} x^2 \frac{|f(x-y)-f(x)|}{|y|}\, dy \leq |x|^2 \sup_{\frac{|x|}{2}<|\xi|<\frac{3}{2}|x|} |f'(\xi)| ,$$

so $I_{1,\varepsilon} \to 0$ uniformly on $\varepsilon>0$ for $|x|\to\infty$ because $f\in \mathcal{S}(\mathbb{R})$.

I_3 It is enough to observe that

$$|y| \geq 2|x| \Rightarrow |x-y| \geq |x-y| - |x| \geq |x|$$

So:

$$|I_3| \leq \int_{|y|\geq 2|x|} |x|^2 \frac{|f(x-y)|}{|y|}\, dy \leq \int_{|x-y|\geq |x|} |x| \, |f(x-y)|\, dy ,$$

which converges to 0 as $|x|\to+\infty$ because $f\in\mathcal{S}(\mathbb{R})$.

I_2 we write

$$\left| I_2 - x\int_{\mathbb{R}} f(x-y)\, dy\right| =$$

$$\leq \left|\int_{\frac{|x|}{2}\leq |y|<2|x|} \left(\frac{x^2}{y} f(x-y) - x f(x-y)\right) dy\right| + \left| x \int_{|y|<\frac{|x|}{2} \vee |y|\geq 2|x|} f(x-y)\, dy\right|$$

$$=: A(x) + B(x).$$

$\boxed{\xi = x-y \text{ \& triangle inequality}}$

Now, $$B(x) \leq |x| \int_{|y|<\frac{|x|}{2} \vee |y|>2|x|} |f(x-y)|\, dy \leq |x| \int_{|\xi|>\frac{|x|}{2}} |f(\xi)|\, d\xi .$$

So, $B(x) \to 0$ as $|x|\to+\infty$ because $f\in\mathcal{S}(\mathbb{R})$.

Finally,

$$|A(x)| \leq |x| \int_{\frac{|x|}{2}\leq |y|<2|x|} \frac{|x-y|}{|y|} |f(x-y)|\, dy \lesssim \int_{\frac{|x|}{2}\leq|z|\leq 2|x|} |z|\,|f(z)|\, dz ,$$

So $|A(x)| \to 0$ as $|x| \to +\infty$ again because $f \in \mathcal{S}(\mathbb{R})$.

Hence, we gather the estimates we performed so far and we obtain:

$$\left| \pi x^2 Hf(x) - x \int_{\mathbb{R}} f(x-y)\, dy \right| \leq \underbrace{|I_{1,\varepsilon}|}_{\to 0} + \underbrace{|A(x)|}_{\to 0} + \underbrace{|B(x)|}_{\to 0} + \underbrace{|I_3|}_{\to 0},$$

for $|x| \to \infty$ / $\varepsilon \to 0$

which proves (*) and, hence, solves the exercise. $\square$

Recall that in the lecture we defined a tempered distribution $T_0 \in \mathcal{S}'(\mathbb{R})$ by

$$\langle T_0, f \rangle := -Hf(0) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(y)}{y}\, dy.$$

Exercise 2 (2 point). Show that the tempered distribution $\widehat{T_0}$ is given by a function, and that $\widehat{T_0}(\xi) = -i\operatorname{sgn}(\xi)$.

Hints:

(i) Let $K_\varepsilon(y) = \frac{1}{y}\mathbf{1}_{|y|>\varepsilon}$, so that $\langle T_0, f\rangle = \frac{1}{\pi}\lim_{\varepsilon\to 0}\langle K_\varepsilon, f\rangle$ for all $f \in \mathcal{S}(\mathbb{R})$. Consider $Q_\varepsilon(y) = \frac{y}{y^2+\varepsilon^2}$ and show that

$$\lim_{\varepsilon \to 0} (K_\varepsilon - Q_\varepsilon) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

that is, $\lim_{\varepsilon\to 0}\langle K_\varepsilon - Q_\varepsilon, f\rangle = 0$ for all $f \in \mathcal{S}(\mathbb{R})$.

(ii) Using the above, justify rigorously that $\widehat{T_0} = \frac{1}{\pi}\lim_{\varepsilon\to 0}\widehat{Q_\varepsilon}$, in the sense of distributions.

(iii) Show that $Q_\varepsilon(x) = \mathcal{F}^{-1}(-\pi i\operatorname{sgn}(\xi)e^{-2\pi\varepsilon|\xi|})(x)$. Conclude that $\widehat{T_0}$ is given by a function, and that $\widehat{T_0}(\xi) = -i\operatorname{sgn}(\xi)$.

Solution.

$\boxed{(i)}$ Let Q_ε and K_ε be as in the statement, and let $f \in \mathcal{S}(\mathbb{R})$.

We write:

$$|\langle K_\varepsilon - Q_\varepsilon, f\rangle| \overset{\text{def}}{=} \left| \int \left(\frac{y}{y^2+\varepsilon^2} - \frac{\mathbb{1}_{|y|>\varepsilon}(y)}{y} \right) f(y)\, dy \right|$$

$$= \left| \int_{|y|>\varepsilon} \left(\frac{y}{y^2+\varepsilon^2} - \frac{1}{y} \right) f(y)\, dy + \int_{|y|\leq\varepsilon} \frac{y}{y^2+\varepsilon^2} f(y)\, dy \right|$$

$$\leq \left| \int_{|y|>\varepsilon} \left(\frac{y}{y^2+\varepsilon^2} - \frac{1}{y} \right) f(y)\, dy \right| + \left| \int_{|y|\leq\varepsilon} \frac{y}{y^2+\varepsilon^2} f(y)\, dy \right| =: ①_\varepsilon + ②_\varepsilon$$

$$\text{①}_\varepsilon = \left| \int_{|y|>\varepsilon} \frac{y^2 - y^2 - \varepsilon^2}{y(y^2+\varepsilon^2)} f(y)\,dy \right| = \varepsilon^2 \int_{|y|>\varepsilon} \frac{1}{y(y^2+\varepsilon^2)} f(y)\,dy$$

$$= \int_{|z|>1} \frac{\cancel{\varepsilon^2}}{z\,\cancel{\varepsilon^2}(z^2+1)} f(z\varepsilon)\,dz \leq \int_{|z|>1} \frac{1}{z^2+1} |f(z\varepsilon)|\,dz$$

change of variables $\boxed{\begin{array}{l} y = \varepsilon z \\ dy = \varepsilon dz \end{array}}$

Hence $\text{①}_\varepsilon \to 0$ as $\varepsilon \to 0$ by dominated convergence theorem and observing that

$$\int_{|z|>1} \frac{1}{z(z^2+1)} f(0)\,dz = 0 \text{ by antisymmetry.}$$

$$\text{②}_\varepsilon = \left| \int_{|y|\leq\varepsilon} \frac{y}{y^2+\varepsilon^2} f(y)\,dy \right| = \left| \int_{|z|\leq 1} \frac{\varepsilon^2 z}{\varepsilon^2 z^2 + \varepsilon^2} f(\varepsilon z)\,dz \right|$$

$\boxed{\begin{array}{l} y = \varepsilon z \\ dy = \varepsilon dz \end{array}}$

$$= \int_{|z|\leq 1} \frac{|z|}{z^2+1} |f(\varepsilon z)|\,dz$$ $\Rightarrow$ we can apply D.C.T. again and obtain that $\text{②}_\varepsilon \to 0$ as $\varepsilon \searrow 0$,

together with the fact that $\int_{|z|\leq 1} \frac{z}{z^2+1} f(0)\,dz = 0$.

$\boxed{\text{ii}}$ Let $f \in \mathcal{S}(\mathbb{R})$. $\langle \widehat{T_0}, f \rangle \overset{\text{def.}}{=} \langle T_0, \hat{f} \rangle = \frac{1}{\pi} \lim_{\varepsilon \to 0} \langle K_\varepsilon, \hat{f} \rangle =$

$$= \frac{1}{\pi} \lim_{\varepsilon\to 0} \langle Q_\varepsilon, \hat{f} \rangle + \cancel{\frac{1}{\pi} \lim_{\varepsilon\to 0} \langle Q_\varepsilon - K_\varepsilon, \hat{f} \rangle}$$ $\boxed{\text{by (i)}}$

$$= \frac{1}{\pi} \lim_{\varepsilon\to 0} \langle Q_\varepsilon, \hat{f} \rangle \overset{\text{def}}{=} \frac{1}{\pi} \lim_{\varepsilon\to 0} \langle \widehat{Q_\varepsilon}, f \rangle .$$

$\boxed{iii}$ We proceed with the calculation.

$$\mathcal{F}^{-1}\left(\cdot\, \pi i \,\mathrm{sgn}(\xi) \exp(-2\pi\varepsilon|\xi|)\right)$$

$$\underset{\text{Four. inversion}}{=} -i\pi \int_{\mathbb{R}} \mathrm{sgn}(\xi) \exp(-2\pi\varepsilon|\xi|) \exp(2\pi i x\cdot\xi)\, d\xi$$

$$\underset{\text{sgn}}{=} \pi i \int_{-\infty}^{0} \exp(2\pi(\varepsilon + ix)\xi)\, d\xi - \pi i \int_0^{+\infty} \exp(2\pi(-\varepsilon + ix)\xi)\, d\xi$$

$$= \pi i \left(\frac{1}{2\pi(\varepsilon + ix)} + \frac{1}{2\pi(-\varepsilon + ix)} \right) = \pi i \frac{2ix}{2\pi(-x^2 - \varepsilon^2)} = \frac{x}{x^2 + \varepsilon^2}$$

$= Q_\varepsilon(x) \implies \mathcal{F}^{-1}(\#)$ is given by a function.

by injectivity of $\mathcal{F}^{-1}$ on $\mathcal{S}'(\mathbb{R})$.

We can conclude observing that $\lim_{\varepsilon \to 0} \widehat{Q_\varepsilon}(\xi) = -i \,\mathrm{sgn}(\xi)$.

$\square$

Recall that an operator $T : \mathcal{S}(\mathbb{R}) \to L^q(\mathbb{R})$ is said to be of strong type (p, q) if there exists a constant $C \in (0, \infty)$ such that $\|Tf\|_{L^q} \leqslant C\|f\|_{L^p}$.

Exercise 3 (1 point). Let $f = \mathbf{1}_{[0,1]}$. Show that for $x \in \mathbb{R} \setminus \{0, 1\}$

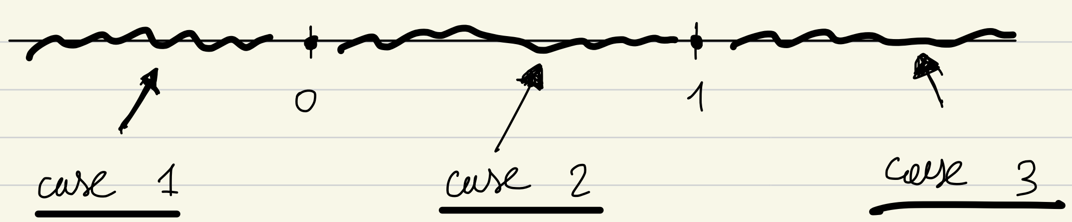
$$\lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy = \log\left|\frac{x}{x-1}\right|.$$

Conclude that the Hilbert transform is neither of strong type (∞, ∞) nor of strong type $(1, 1)$.

Solution. For $\varepsilon \in (0,1)$ and $x \neq 0,1$, and define

$$F_\varepsilon(x) := \int_{|x-y|>\varepsilon} \frac{\chi_{[0,1]}}{x-y}\, dy \quad \text{and} \quad F(x) := \log\left|\frac{x}{x-1}\right|.$$

Now, we split cases depending on the position of x.



Case 1. Assume that $x \in (0, \infty)$ and that $\varepsilon \in (0, |x|)$.

We have

$$F_\varepsilon(x) = \int_{|y-x|>\varepsilon} \frac{\chi_{[0,1]}(y)}{x-y}\, dy = \int_0^1 \frac{1}{x-y}\, dy = F(x) \quad \forall\, \varepsilon \in (0, |x|).$$

Case 2. Assume that $x \in (0,1)$, and that $\varepsilon > 0$ is s.t. $[x-\varepsilon, x+\varepsilon] \subset [0,1]$.

It holds:

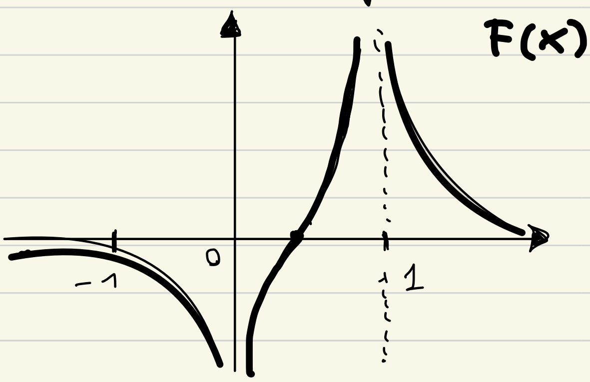
$$F_\varepsilon(x) = \int_{|y-x|>\varepsilon} \frac{\chi_{[0,1]}(y)}{x-y}\, dy = \int_0^{x-\varepsilon} \frac{1}{x-y}\, dy + \int_{x+\varepsilon}^1 \frac{1}{x-y}\, dy$$

$$= \log\frac{|x|}{\varepsilon} + \log\frac{\varepsilon}{|x-1|} = \log\frac{|x|}{|x-1|} = F(x) \quad \forall\, \varepsilon \text{ as above.}$$

Case 3. Analogous to Case 1.

Finally, we notice that:

$$\log\left|\frac{x}{x-1}\right| = \log|x| - \log|x-1|$$



It's evident that $F \notin L^\infty(\mathbb{R})$.

- Let's show that $F \notin L^1(\mathbb{R})$. For $|x| \leq \frac{1}{10}$, we have

$$|F(x)| \geq \Big|\,|\log|x|| - \underbrace{|\log|x-1||}_{\leq C}\,\Big| \geq |\log|x|| - C \Rightarrow F \notin L^1(\mathbb{R}).$$

$\leq C$ (local smoothness of $\log|x-1|$ away from $x=1$)

$\square$

Recall that the essential support of a locally integrable function f is the smallest closed set, denoted by $\operatorname{ess\,supp}(f)$, such that $f = 0$ a.e. on the complement of $\operatorname{ess\,supp}(f)$.

Exercise 4 (1 point)**.** Show that if $f \in L^2(\mathbb{R})$, then for a.e. $x \notin \operatorname{ess\,supp}(f)$

$$Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

Hint: You may use the fact that for $f_n(x) := (f\mathbf{1}_{B(0,n)}) * \varphi_{1/n}(x) \in C_c^\infty(\mathbb{R})$ we have $f_n \to f$ in $L^2(\mathbb{R})$. Here φ_ε is a smooth mollifier: $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$, $\varphi \in C_c^\infty(\mathbb{R})$, $\mathbf{1}_{[-0.1,0.1]} \leqslant \varphi \leqslant \mathbf{1}_{[-1,1]}$, and $\|\varphi\|_{L^1} = 1$.

Solution. Let $f \in L^2(\mathbb{R})$, and f_n be the function defined above.

Observe that $f_n \in C_c^\infty(\mathbb{R}) \Rightarrow f_n \in \mathcal{S}(\mathbb{R})$ $\forall$ n.

Moreover,

$$\| H(f_n - f) \|_{L^2(\mathbb{R})} = \| f_n - f \|_{L^2(\mathbb{R})}$$

$\Rightarrow$ possibly by passing to a subsequence we have

$$Hf_n(x) \to Hf(x) \quad \text{for a.e. } x \in \mathbb{R}.$$

Moreover, $\forall$ $x \notin \text{ess-supp}(f)$ $\exists$ $\overline{N}(x) > 0$ such that

$$\operatorname{dist}(x, \operatorname{supp}(f_n)) \geq \frac{\operatorname{dist}(x, \text{ess-supp}(f))}{2} =: \delta \quad \forall \quad n \geq \overline{N}(x).$$

For such values of n it holds

$$\left| Hf_n(x) - \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy \right| \overset{f_n \in \mathcal{S}(\mathbb{R})}{=} \frac{1}{\pi} \left| \int_{\mathbb{R}} \left(\frac{f_n(y)}{x-y} - \frac{f(y)}{x-y} \right) dy \right|$$

$$= \frac{1}{\pi} \left| \int_{|x-y| \geq \delta} \frac{1}{(x-y)} (f_n(y) - f(y))\, dy \right|$$

$$\leq \frac{1}{\pi} C(\delta) \| f_n - f \|_{L^2(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$

Cauchy-Schwarz

Hence $Hf(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy$

for a.e. $x \notin \text{ess-supp}(f)$. $\square$