SINGULAR INTEGRAL OPERATORS
EXERCISE IV (21.11.2023)

Exercise 1 (1 point)**.** Let $1 \leqslant p < \infty$ and $w \in A_p$. Show that $L^1(\mathbb{R}^n)$ is dense in $L^p(w)$.

Hint: For any $f \in L^p(w)$ prove that $f_R := f\mathbf{1}_{B(0,R)} \in L^1(\mathbb{R}^n)$ for all $R > 0$, and that $f_R \to f$ in $L^p(w)$ as $R \to \infty$. The estimate (6.9) from lecture notes (and its modification for $p = 1$) may be helpful.

Exercise 2 (1 point)**.** Prove that in the definition of the A_p condition we may replace cubes by balls and still get the same class of weights. More specifically,

$$\sup_{Q \subset \mathbb{R}^n} \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} \sim \sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B w \right) \left(\frac{1}{|B|} \int_B w^{1-p'} \right)^{p-1},$$

where Q are cubes and B are balls.

Hint: The doubling condition ($w(2B) \leqslant Cw(B)$ for all balls) is relevant.

Exercise 3 (2 points)**.** Prove that $w(x) = |x|^a$ is an A_p weight on $\mathbb{R}^n$, $1 < p < \infty$, if and only if $-n < a < n(p-1)$.

Hint: Show first that $w(x) = |x|^a$ is a doubling weight if and only if $a > -n$. Consider separately balls $B = B(x_0, r)$ such that $|x_0| \geqslant 3r$, and such that $|x_0| < 3r$.

Exercise 4 (1 point)**.** Show that

$$w(x) = \begin{cases} \log \frac{1}{|x|} & |x| \leqslant e^{-1} \\ 1 & |x| > e^{-1} \end{cases}$$

is an A_1 weight on $\mathbb{R}^n$.