SINGULAR INTEGRAL OPERATORS
EXERCISE III (14.11.2023)

Exercise 1 (1 point)**.** Prove that if a Calderón-Zygmund operator T is associated to a standard kernel K, then its adjoint (see Section 4.3 in the notes) is also a Calderón-Zygmund operator, and it is associated to the standard kernel

$$K^t(x,y) = \overline{K(y,x)}.$$

The following exercise demonstrates that the strong type $(2,2)$ estimate in the definition of Calderón-Zygmund operators can be replaced by any other strong type (p,p) estimate, $1 < p < \infty$, and the class of operators remains the same.

Exercise 2 (1 point)**.** Suppose that in the definition of Calderón-Zygmund operators (Definition 4.9) we replaced the strong type $(2,2)$ estimate by the strong type (p,p) estimate for some $1 < p < \infty$. How would you modify the proof of Theorem 4.13 to get that these "new" Calderón-Zygmund operators are weak type $(1,1)$ and strong type (q,q), $1 < q < \infty$?[1] You don't have to repeat the whole proof, just the parts that change compared to $p = 2$.

Exercise 3 (2 points)**.** Let T be a Calderón-Zygmund singular integral operator, and T_ε the associated truncated operators. Prove that for $f \in L^1(\mathbb{R}^n)$

$$\lim_{\varepsilon \to 0} T_\varepsilon f(x) = Tf(x) \qquad \text{for a.e. } x \in \mathbb{R}^n, \tag{0.1}$$

and for $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$\lim_{\varepsilon \to 0} \|T_\varepsilon f - Tf\|_{L^p} = 0. \tag{0.2}$$

Hint: For (0.1) use the weak type $(1,1)$ estimates of T and the maximal operator T_*. For (0.2) use that $\lim_{\varepsilon \to 0} T_\varepsilon f(x) = Tf(x)$ for a.e. $x \in \mathbb{R}^n$ (shown in the lecture) and the dominated convergence theorem.

Recall that M is the usual Hardy-Littlewood maximal operator, and its modified version M_c is defined as

$$M_c f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f|,$$

where the supremum is taken over all axis-parallel cubes containing x.

Exercise 4 (1 point)**.** Show that there exists $C = C(n) \geqslant 1$ such that for any $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have

$$C^{-1} Mf(x) \leqslant M_c f(x) \leqslant C Mf(x).$$

Conclude that M is of weak type (p,p) with respect to a weight w for some $1 \leqslant p < \infty$ if and only if M_c is of weak type (p,p) with respect to w.

Exercise 5 (1 point)**.** Show that the A_1 condition is equivalent to

$$M_c w(x) \leqslant C w(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

[1] In particular, the "new" definition of CZ operators is equivalent to the one from the lectures.