SINGULAR INTEGRAL OPERATORS
EXERCISE II (7.11.2023)

Exercise 1 (1 point)**.** Show that for every Hölder continuous $\Omega : \mathbb{S}^{n-1} \to \mathbb{C}$ the kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ defined by

$$K(x,y) = \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n}$$

is a standard kernel.

Exercise 2 (1 point)**.** Prove that if A is Lipschitz, then the Cauchy kernel

$$K(x,y) = \frac{1}{x - y + i(A(x) - A(y))}$$

is a standard kernel with $\delta = 1$.

Exercise 3 (2 points)**.** If T is a Calderón-Zygmund operator such that it is associated with two kernels K_1 and K_2, that is, for all $f \in L^2(\mathbb{R}^n)$ with compact support

$$Tf(x) = \int K_1(x,y) f(y)\, dy = \int K_2(x,y) f(y)\, dy \qquad \text{for } x \notin \operatorname{supp} f,$$

then $K_1 = K_2$ a.e.

Hint: Assume that the claim is false. You should find a positive measure set $E \subset \mathbb{R}^n$ and a point $x \notin E$ such that $K_1(x,y) - K_2(x,y)$ has a fixed sign for $y \in E$.

Recall that $\mathcal{D}(\mathbb{R}^n)$ denotes the family of dyadic cubes. The notation $A \lesssim B$ stands for "there exists a dimensional constant $C \geqslant 1$ such that $A \leqslant CB$," and $A \sim B$ means $A \lesssim B \lesssim A$. Given $Q \in \mathcal{D}(\mathbb{R}^n)$ we write CQ to denote the cube with the same center as Q and with sidelength $C\ell(Q)$.

Exercise 4 (2 point)**.** Suppose that $\Omega \subsetneq \mathbb{R}^n$ is an open set. Let $\mathcal{W} \subset \mathcal{D}(\mathbb{R}^n)$ be the family of maximal[1] cubes contained in Ω and satisfying $10Q \cap \Omega^c = \varnothing$. Prove that

(i) the cubes in $\mathcal{W}$ are pairwise disjoint, and $\bigcup_{Q \in \mathcal{W}} Q = \Omega$,
(ii) for every $Q \in \mathcal{W}$ we have $\ell(Q) \sim \operatorname{dist}(Q, \Omega^c)$,
(iii) for every $P, Q \in \mathcal{W}$ with $3P \cap 3Q \neq \varnothing$ we have $\ell(P) \sim \ell(Q)$.
(iv) for every $Q \in \mathcal{W}$ we have $\#\{P \in \mathcal{W} : 3P \cap 3Q \neq \varnothing\} \lesssim 1$.

The family $\mathcal{W}$ is called *the Whitney decomposition* of Ω, and it has many applications in analysis.

[1] maximal with respect to inclusion