SINGULAR INTEGRAL OPERATORS
EXERCISE I (31.10.2023)

Exercise 1 (1 point)**.** Let $f \in \mathcal{S}(\mathbb{R})$. Show that $Hf \in L^1(\mathbb{R})$ if and only if $\int_{\mathbb{R}} f(y)\, dy = 0$.
Hint: Modify the proof of Lemma 3.5 from the lecture notes to estimate the asymptotics of $x^2 \cdot Hf(x)$ as $|x| \to \infty$.

Recall that in the lecture we defined a tempered distribution $T_0 \in \mathcal{S}'(\mathbb{R})$ by

$$\langle T_0, f\rangle := -Hf(0) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(y)}{y}\, dy.$$

Exercise 2 (2 point)**.** Show that the tempered distribution $\widehat{T_0}$ is given by a function, and that $\widehat{T_0}(\xi) = -i \operatorname{sgn}(\xi)$.
Hints:

(i) Let $K_\varepsilon(y) = \frac{1}{y}\mathbf{1}_{|y|>\varepsilon}$, so that $\langle T_0, f\rangle = \frac{1}{\pi}\lim_{\varepsilon\to 0}\langle K_\varepsilon, f\rangle$ for all $f \in \mathcal{S}(\mathbb{R})$. Consider $Q_\varepsilon(y) = \frac{y}{y^2+\varepsilon^2}$ and show that

$$\lim_{\varepsilon\to 0} (K_\varepsilon - Q_\varepsilon) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

that is, $\lim_{\varepsilon\to 0} \langle K_\varepsilon - Q_\varepsilon, f\rangle = 0$ for all $f \in \mathcal{S}(\mathbb{R})$.

(ii) Using the above, justify rigorously that $\widehat{T_0} = \frac{1}{\pi}\lim_{\varepsilon\to 0} \widehat{Q_\varepsilon}$, in the sense of distributions.

(iii) Show that $Q_\varepsilon(x) = \mathcal{F}^{-1}(-\pi i \operatorname{sgn}(\xi) e^{-2\pi\varepsilon|\xi|})(x)$. Conclude that $\widehat{T_0}$ is given by a function, and that $\widehat{T_0}(\xi) = -i\operatorname{sgn}(\xi)$.

Recall that an operator $T : \mathcal{S}(\mathbb{R}) \to L^q(\mathbb{R})$ is said to be of strong type (p, q) if there exists a constant $C \in (0, \infty)$ such that $\|Tf\|_{L^q} \leqslant C\|f\|_{L^p}$.

Exercise 3 (1 point)**.** Let $f = \mathbf{1}_{[0,1]}$. Show that for $x \in \mathbb{R}\backslash\{0, 1\}$

$$\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy = \log\left|\frac{x}{x-1}\right|.$$

Conclude that the Hilbert transform is neither of strong type (∞, ∞) nor of strong type $(1, 1)$.

Recall that the essential support of a locally integrable function f is the smallest closed set, denoted by $\operatorname{ess\,supp}(f)$, such that $f = 0$ a.e. on the complement of $\operatorname{ess\,supp}(f)$.

Exercise 4 (1 point)**.** Show that if $f \in L^2(\mathbb{R})$, then for a.e. $x \notin \operatorname{ess\,supp}(f)$

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

Hint: You may use the fact that for $f_n(x) := (f\mathbf{1}_{B(0,n)}) * \varphi_{1/n}(x) \in C_c^\infty(\mathbb{R})$ we have $f_n \to f$ in $L^2(\mathbb{R})$. Here φ_ε is a smooth mollifier: $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(x/\varepsilon)$, $\varphi \in C_c^\infty(\mathbb{R})$, $\mathbf{1}_{[-0.1,0.1]} \leqslant \varphi \leqslant \mathbf{1}_{[-1,1]}$, and $\|\varphi\|_{L^1} = 1$.