Cones, rectifiability, and SIOs

Damian Dąbrowski

Universitat Autònoma de Barcelona, Barcelona Graduate School of Mathematics (BGS-Math)

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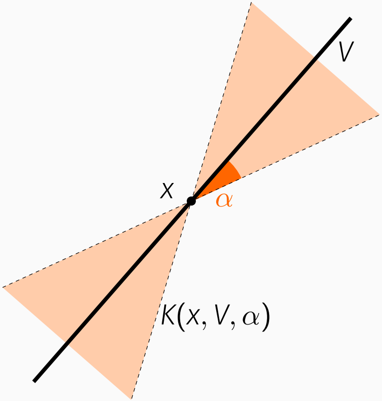
Cones

Given $x \in \mathbb{R}^d$, $V \in G(d,m)$, $\alpha \in (0,1)$, set

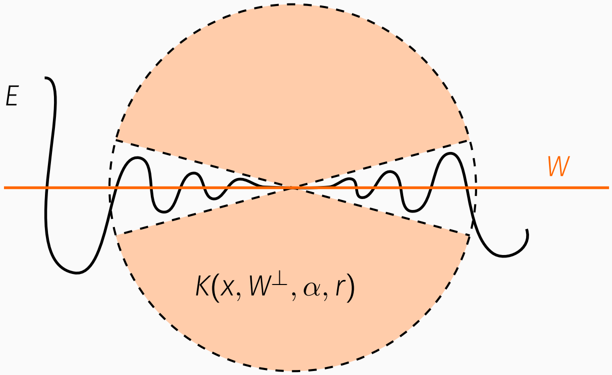
$$K(x,V,\alpha) = \{y \in \mathbb{R}^d \;:\; \operatorname{dist}(y, x+V) < \alpha|y-x|\},$$
$$K(x,V,\alpha,r) = K(x,V,\alpha) \cap B(x,r).$$



Tangent planes

A plane $W \in G(d, n)$ is a **tangent plane** to E at x if for all $\alpha \in (0, 1)$ there exists $r > 0$ such that

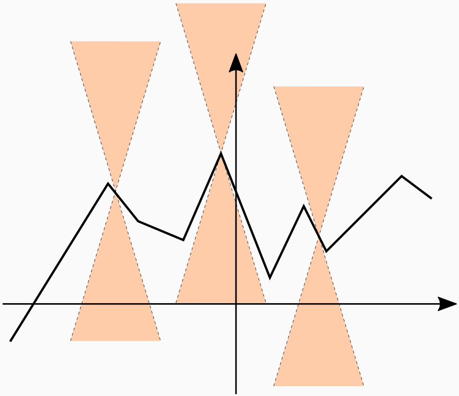
$$E \cap K(x, W^{\perp}, \alpha, r) = \varnothing.$$



Cones and Lipschitz graphs

Easy to show: $E \subset \mathbb{R}^d$ is a subset of an n-dimensional Lipschitz graph iff there exists $V \in G(d, d-n)$, $\alpha \in (0,1)$, such that

$$x \in E \quad \Rightarrow \quad E \cap K(x, V, \alpha) = \varnothing.$$



Rectifiability

Rectifiable sets and measures

A set $E \subset \mathbb{R}^d$ is *n*-**rectifiable** if there exists a countable number of n-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^n\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

A measure μ on $\mathbb{R}^d$ is *n*-**rectifiable** if it is of the form

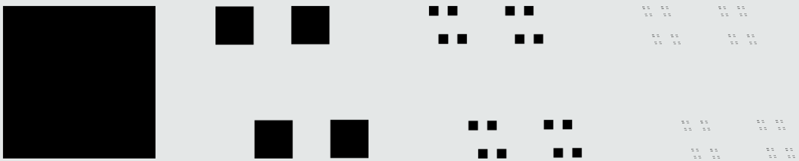
$$\mu = f\mathcal{H}^n|_E$$

for some n-rectifiable $E \subset \mathbb{R}^d$ and $f \in L^1_{loc}(E)$.

We say that $F \subset \mathbb{R}^d$ is **purely n-unrectifiable** if for every Γ - Lipschitz image of $\mathbb{R}^n$

$$\mathcal{H}^n(F \cap \Gamma) = 0.$$

Example



F is purely 1-unrectifiable and satisfies $1 \leq \mathcal{H}^1(F) \leq \sqrt{2}$.

Any set of finite $\mathcal{H}^n$ measure can be decomposed into a rectifiable and purely unrectifiable part.

Why do we care?

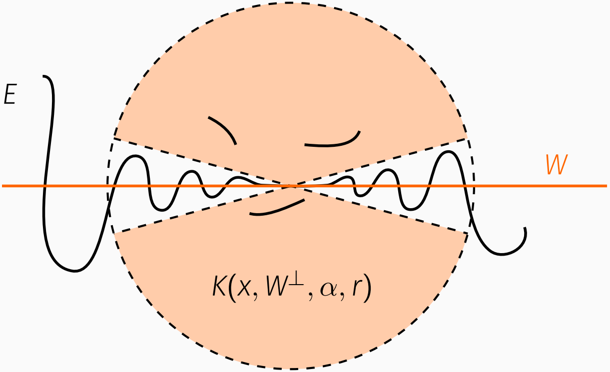
Applications in:

- boundedness of singular integral operators,
- study of removable sets for bounded analytic functions,
- optimal regularity of domains that ensure L^p solvability of the Dirichlet problem,
- study of singular sets of harmonic maps, free boundaries...

Approximate tangent planes

A plane $W \in G(d,n)$ is an approximate **tangent plane** to E at x if for all $\alpha \in (0,1)$

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap K(x, W^\perp, \alpha, r))}{r^n} = 0.$$



Approximate tangents characterize rectifiability

Theorem (Federer '47)

Let $E \subset \mathbb{R}^d$, $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable iff for $\mathcal{H}^n$-a.e. $x \in E$ there is a unique approximate tangent plane to E at x, i.e. for all α

$$\lim_{r \to 0} \frac{\mathcal{H}^n(E \cap K(x, W^\perp, \alpha, r))}{r^n} = 0.$$

Analogous result holds for μ satisfying $0 < \Theta^{n,*}(\mu, x) < \infty$,

$$\Theta^{n,*}(\mu, x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{r^n}, \qquad \Theta^{n}_{*}(\mu, x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{r^n}.$$

Fact

$$\mu \text{ is rectifiable} \quad \Rightarrow \quad 0 < \Theta^{n,*}(\mu, x) = \Theta^{n}_{*}(\mu, x) < \infty \text{ a.e.}$$

Conical energy

Let $V \in G(d, d-n),\ \alpha \in (0,1),\ 1 \leq p < \infty$. The (V, α, p) **conical energy** of E at $x \in E$ up to scale $R > 0$ is

$$\mathcal{E}_{E,p}(x, V, \alpha, R) = \int_0^R \left(\frac{\mathcal{H}^n(E \cap K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r}.$$

More generally: for a Radon measure μ on $\mathbb{R}^d$ define

$$\mathcal{E}_{\mu,p}(x, V, \alpha, R) = \int_0^R \left(\frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r}.$$

Rectifiability implies finite energy

Theorem (D. '20)

Let $1 \leq p < \infty$. Suppose μ is n-rectifiable. Then, for μ-a.e. x there is $V_x \in G(d, d-n)$ such that for all $\alpha \in (0,1)$

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) = \int_0^1 \left(\frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Finite energy implies rectifiability

Theorem (D. '20)

Let $1 \leq p < \infty$. Suppose μ is a Radon measure satisfying $0 < \Theta^{n,*}(\mu, x)$ and $\Theta^n_*(\mu, x) < \infty$. Assume that for μ-a.e. x there is $V_x \in G(d, d-n)$ and $\alpha \in (0,1)$ such that

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) = \int_0^1 \left(\frac{\mu(K(x, V_x, \alpha, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Then, μ is n-rectifiable.

Question

$0 < \Theta^{n,*}(\mu, x),\ \Theta^n_*(\mu, x) < \infty,$ approximate tangents exist a.e. $\stackrel{?}{\Longrightarrow}$ μ is rectifiable

Remarks on the proofs

μ is rectifiable $\Rightarrow$ $\mathcal{E}_{\mu,p} < \infty$ a.e.

Follows easily from a result of Tolsa:

Theorem (Tolsa '15)

μ is rectifiable $\Rightarrow$ $\int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty$ a.e.

Not difficult:

$$\mathcal{E}_{\mu,1}(x,V,\alpha,1) = \int_0^1 \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{dr}{r} \lesssim \int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty,$$

and

$$\mathcal{E}_{\mu,p}(x,V,\alpha,1) \leq \Theta^{n,*}(\mu,x)^{p-1} \, \mathcal{E}_{\mu,1}(x,V,\alpha,1).$$

■

Remarks on the proofs

$$\begin{array}{c} 0 < \Theta^{n,*}(\mu,x),\ \Theta^{n}_{*}(\mu,x) < \infty, \\ \mathcal{E}_{\mu,p} < \infty \text{ a.e.} \end{array} \implies \mu \text{ is rectifiable}$$

- a corona decomposition result,
- prove the theorem assuming additionally $\Theta^{n,*}(\mu,x) < \infty$,
- show that

$$\begin{array}{c} 0 < \Theta^{n,*}(\mu,x),\ \Theta^{n}_{*}(\mu,x) < \infty, \\ \mathcal{E}_{\mu,p} < \infty \text{ a.e.} \end{array} \implies \Theta^{n,*}(\mu,x) < \infty.$$

■

Big pieces of Lipschitz graphs

Big pieces of Lipschitz graphs

We say that $E \subset \mathbb{R}^d$ has **big pieces of Lipschitz graphs** (BPLG) if there exists $C, L, \kappa > 0$ such that

- it is AD-regular, i.e. for $x \in E,\ 0 < r < \operatorname{diam}(E)$

$$C^{-1} r^n \leq \mathcal{H}^n(E \cap B(x,r)) \leq C r^n,$$

- for all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$, there exists a Lipschitz graph Γ, $\operatorname{Lip}(\Gamma) \leq L$, such that

$$\mathcal{H}^n(E \cap B \cap \Gamma) \geq \kappa r(B)^n.$$

Big pieces of Lipschitz graphs

Characterizing BPLG using conical energy

Theorem (D. '20)

Suppose $E \subset \mathbb{R}^d$ is AD-regular, $1 \leq p < \infty$. Then E has BPLG iff there exist $\alpha, \kappa, M > 0$, such that the following holds.

For all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$, there exists a set $G_B \subset E \cap B$ with $\mathcal{H}^n(G_B) \geq \kappa\, r(B)^n$, and a direction $V \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{E,p}(x, V, \alpha, r(B)) = \int_0^{r(B)} \left(\frac{\mathcal{H}^n(E \cap K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \leq M.$$

We will call the condition above **big pieces with bounded energy** (BPBE).

Proof of "E has BPBE $\Rightarrow$ E has BPLG"

E has BPBE $\quad\Rightarrow\quad$ E has BPLG

Can be reduced to

Theorem (Martikainen-Orponen '18)

Suppose $E \subset \mathbb{R}^d$ is AD-regular. Then E has BPLG iff there exist $\kappa, M > 0$, such that the following holds.

For all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$, there exists a set $G_B \subset E \cap B$ with $\mathcal{H}^n(G_B) \geq \kappa\, r(B)^n$, and a direction $V_B \in G(d,n)$, such that for a.e. $W \in \mathbf{B}(V_B, \kappa)$ we have $(\pi_W)_*(\mathcal{H}^n|_{G_B}) \in L^2(W)$, and

$$\int_{\mathbf{B}(V_B,\kappa)} \|(\pi_W)_*(\mathcal{H}^n|_{G_B})\|^2_{L^2(W)}\, d\gamma_{d,n}(W) \leq M r(B)^n.$$

Bounded mean energy condition

Definition

We will say that an AD-regular set E satisfies the **bounded mean energy** condition if there exist $\alpha > 0, M > 1$, and for a.e. $x \in E$ there exists $V_x \in G(d, d-n)$, such that:

for all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$,

$$\int_{E\cap B} \mathcal{E}_{E,p}(x, V_x, \alpha, r(B))\, d\mathcal{H}^n(x) = \int_{E\cap B}\int_0^{r(B)} \left(\frac{\mathcal{H}^n(E\cap K(x, V_x, \alpha, r))}{r^n}\right)^p \frac{dr}{r} d\mathcal{H}^n(x) \leq M r(B)^n.$$

Easy: BME $\Rightarrow$ BPBE. In particular, BME $\Rightarrow$ BPLG. But the converse is not true!

Question

How to modify BME to get a characterization of BPLG or UR? Replace V_x by $V_{x,r}$?

Singular integral operators

Boundedness of singular integral operators

Given a Radon measure μ, $f \in L^2(\mu)$, a kernel $K(x,y)$, and $\varepsilon > 0$ set

$$T_{\mu,\varepsilon}f(x) = \int_{|x-y|>\varepsilon} K(x,y)f(y)\, d\mu(y).$$

We say that T_μ is bounded on $L^2(\mu)$ if $\|T_{\mu,\varepsilon}\|_{L^2(\mu)\to L^2(\mu)}$ are bounded uniformly in ε.

Examples

- Cauchy transform $\mathcal{C}_\mu f(z) = \int_{\mathbb{C}} \frac{f(w)}{z-w}\, d\mu(w)$,
- n-dimensional Riesz transform $\mathcal{R}_\mu f(x) = \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y)$.

Question

Given a “nice” kernel K, what are the measures μ such that T_μ is bounded on $L^2(\mu)$?

Denote by $\mathcal{K}^n(\mathbb{R}^d)$ the class of kernels of the form $K(x,y) = k(x-y)$, where $k : \mathbb{R}^d \to \mathbb{R}$ are smooth, odd, and satisfy

$$|\nabla^j k(x)| \leq C_j |x|^{-n-j}, \quad j = 0, 1, 2, \ldots$$

Theorem (David-Semmes ’91)

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

for all $K \in \mathcal{K}^n(\mathbb{R}^d)$ T_μ is bounded on $L^2(\mu)$ $\Leftrightarrow$ μ is uniformly rectifiable.

David-Semmes conjecture

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

$$\mathcal{R}_\mu \text{ is bounded on } L^2(\mu) \quad \Leftrightarrow \quad \mu \text{ is uniformly rectifiable.}$$

True for $n = 1$ (Mattila-Melnikov-Verdera 1996) and $n = d - 1$ (Nazarov-Tolsa-Volberg 2012).

Question

If we only assume that $\mu(B(x, r)) \leq Cr^n$, what are the necessary/sufficient conditions for boundedness of $\mathcal{R}_\mu$?

BPBE with $p = 1$ and SIOs

Theorem (Chang-Tolsa '17)

Let μ be a Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$.
Suppose that μ satisfies the BPBE conditions with $p = 1$, i.e.
there exist constants $\alpha, \kappa, M > 0$, such that:
for all balls B there exists a set $G_B \subset B$ with $\mu(G_B) \geq \kappa\, \mu(B)$,
and a direction $V_B \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{\mu,1}(x, V_B, \alpha, r(B)) \leq M.$$

Then, for all $K \in \mathcal{K}^n(\mathbb{R}^d)$ we have T_μ bounded on $L^2(\mu)$.

BPBE with $p = 2$ and SIOs

Theorem (D. '20)

Let μ be a Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$.
Suppose that μ satisfies the BPBE conditions with $p = 2$, i.e.
there exist constants $\alpha, \kappa, M > 0$, such that:
for all balls B there exists a set $G_B \subset B$ with $\mu(G_B) \geq \kappa\, \mu(B)$,
and a direction $V_B \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{\mu,2}(x, V_B, \alpha, r(B)) \leq M.$$

Then, for all $K \in \mathcal{K}^n(\mathbb{R}^d)$ we have T_μ bounded on $L^2(\mu)$.

This is strictly stronger than the result of Chang and Tolsa:

$$\int_0^R \left(\frac{\mu(K(x,V,\alpha,r))}{r^n}\right)^2 \frac{dr}{r} \leq \int_0^R \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{\mu(B(x,r))}{r^n} \frac{dr}{r} \leq C \int_0^R \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{dr}{r}.$$

Corona decomposition

Main lemma

Let μ be a compactly supported Radon measure on $\mathbb{R}^d$ satisfying $\mu(B(x,r)) \leq Cr^n$. Assume further that for some $V \in G(d, d-n)$, $\alpha \in (0,1)$, we have

$$\mathcal{E}_{\mu,p}(\mathbb{R}^d) = \int \mathcal{E}_{\mu,p}(x, V, \alpha, \infty)\, d\mu(x) < \infty.$$

Then, there exists a decomposition $\mathcal{D}_\mu = \bigcup_{R \in \mathrm{Top}} \mathrm{Tree}(R)$, and a corresponding family of Lipschitz graphs $\{\Gamma_R\}_{R \in \mathrm{Top}}$, satisfying:

(i) Lipschitz constants of Γ_R are uniformly bounded,

(ii) μ-almost all of $R \setminus \bigcup_{Q \in \mathrm{Stop}(R)} Q$ is contained in Γ_R,

(iii) for all $Q \in \mathrm{Tree}(R)$ we have $\Theta_\mu(2B_Q) \lesssim \Theta_\mu(2B_R)$

(iv) we have the packing condition

$$\sum_{R \in \mathrm{Top}} \Theta_\mu(2B_R)^p \mu(R) \lesssim \mu(\mathbb{R}^d) + \mathcal{E}_{\mu,p}(\mathbb{R}^d).$$