On measures, projections, and measures of projections

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On measures

Let $\mathcal{B}$ be the family of Borel subsets of $\mathbb{R}^2$. We say that $\mu : \mathcal{B} \to [0, \infty]$ is a (Borel) measure if

- $\mu(\varnothing) = 0$,
- for all countable families $\{E_k\}$ of disjoint sets

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- the arc-length measure on a smooth curve.

Hausdorff measure

Let $0 \leq s \leq 2$. For $E \subset \mathbb{R}^2$ and $0 < \delta < \infty$ we define

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_i \operatorname{diam}(A_i)^s \ : \ E \subset \bigcup_i A_i, \ \operatorname{diam}(A_i) \leq \delta \right\}.$$

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$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E).$$

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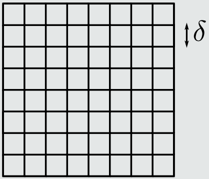

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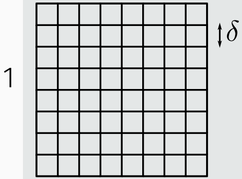

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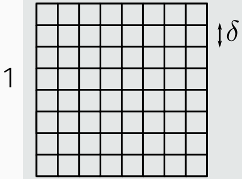

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$$\begin{aligned} \mathcal{H}^s(E) &= 0 && \text{for } 1 < s \le 2 \\ 0 < \mathcal{H}^1(E) &< \infty && \text{for } s = 1 \\ \mathcal{H}^s(E) &= \infty && \text{for } s < 1 \end{aligned}$$

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Fun fact

$\mathcal{H}^1 = c\mathcal{L}^1, \ \mathcal{H}^2 = c\mathcal{L}^2!$

Hausdorff dimension

Fact

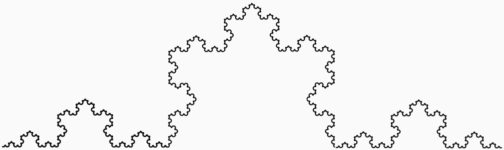
For any Borel set $E \subset \mathbb{R}^2$ there exists a unique $0 \leq s_0 \leq 2$ such that

$$\mathcal{H}^s(E) = 0 \quad \text{for } s_0 < s \leq 2$$
$$\mathcal{H}^s(E) = \infty \quad \text{for } 0 \leq s < s_0.$$

We call such s_0 the Hausdorff dimension of E, and we denote it by $\dim(E)$.

Hausdorff dimension - examples

$$s_0 = \log_3(2)$$



$$s_0 = \log_3(4)$$

Hausdorff measures and Lipschitz maps

We say that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is L-Lipschitz if for any $x, y \in \mathbb{R}^2$

$$|f(x) - f(y)| \leq L|x - y|.$$

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Proof: Note that for any covering A_i of E, the family $f(A_i)$ covers $f(E)$, and moreover $\operatorname{diam}(f(A_i)) \leq L \operatorname{diam}(A_i)$. Hence

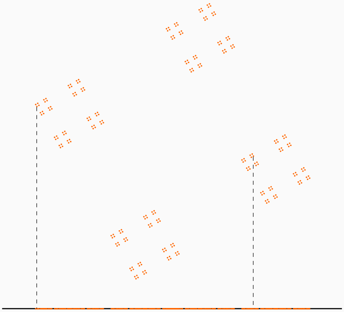
$$\sum_i \operatorname{diam}(f(A_i))^s \leq \sum_i (L \operatorname{diam}(A_i))^s = L^s \sum_i \operatorname{diam}(A_i)^s.$$

Projections

Given a line $L \subset \mathbb{R}^2$ we will denote the orthogonal projection onto L by π_L.

Question

Given $E \subset \mathbb{R}^2$ what is the relation between $\dim(E)$ and $\dim(\pi_L(E))$?



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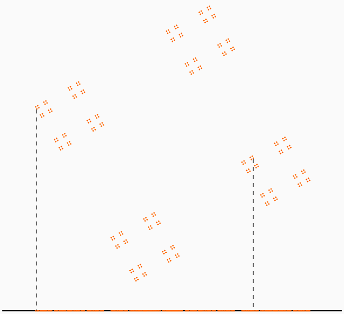

Note that π_L is 1-Lipschitz, and so

$$\mathcal{H}^s(\pi_L(E)) \leq \mathcal{H}^s(E).$$

In consequence,

$$\dim(\pi_L(E)) \leq \dim(E).$$

Marstrand's projection theorem

Theorem (Marstrand 1954)

If $\dim(E) \leq 1$, then for almost every line L

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Question 1

What is the dimension of the exceptional set of lines, i.e. lines L such that $\dim(\pi_L(E)) < \dim(E)$?

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If $\dim(E) > 1$, then for almost every line L

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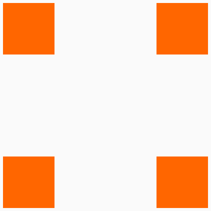
Question 2

If $\dim(E) = 1$, and $0 < \mathcal{H}^1(E) < \infty$, do we have for almost every line L

$$\mathcal{H}^1(\pi_L(E)) > 0?$$

Four-corner Cantor set

The answer is no! There exists a set K with $1 \leq \mathcal{H}^1(K) \leq \sqrt{2}$ that projects to a zero length set in almost every direction.



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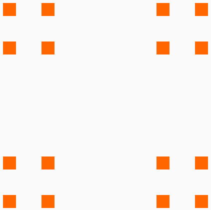

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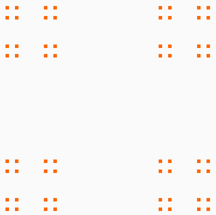

K_3

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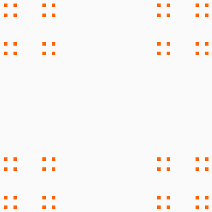

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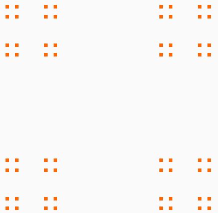
$$\mathcal{H}^1_{4^{-n+1}}(K) = \inf\{\textstyle\sum_i \operatorname{diam}(A_i)^1\} \leq 4^n(4^{-n}\sqrt{2})^1 = \sqrt{2}$$



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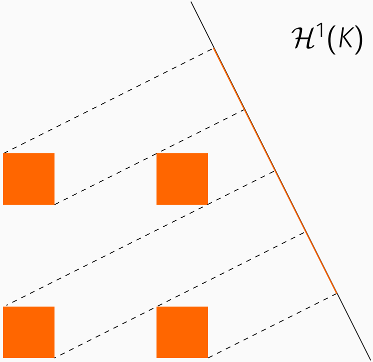

$$\mathcal{H}^1(K) \geq \mathcal{H}^1(\pi_L(K)) \geq 1$$

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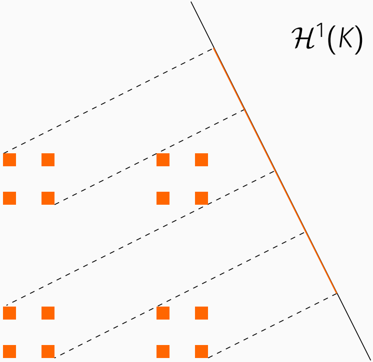

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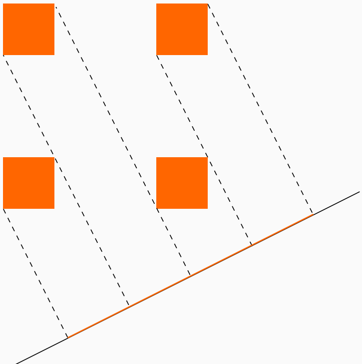

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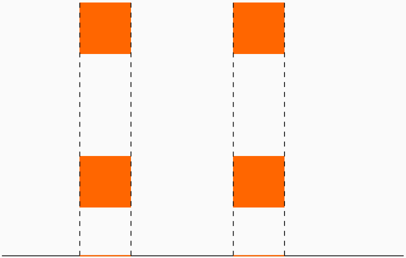
$$\mathcal{H}^1(\pi_{L'}(K_1)) = \frac{1}{2} \quad \Rightarrow \quad \mathcal{H}^1(\pi_{L'}(K_n)) = \frac{1}{2^n}$$



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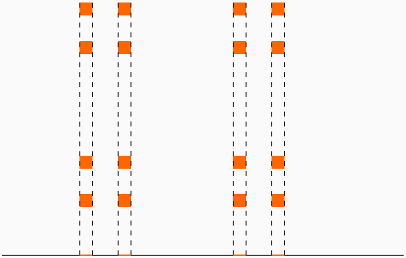

Rectifiable sets

A set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of 1-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^1\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

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We say that $F \subset \mathbb{R}^2$ is purely unrectifiable if for every Lipschitz graph Γ

$$\mathcal{H}^1(F \cap \Gamma) = 0.$$

Any set of finite $\mathcal{H}^1$ measure can be decomposed into a rectifiable and purely unrectifiable part.

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. E is purely unrectifiable iff

$$\mathcal{H}^1(\pi_L(E)) = 0 \quad \text{for a.e. } L.$$

In particular, if E is rectifiable with $0 < \mathcal{H}^1(E) < \infty$, then for almost every line

$$\mathcal{H}^1(\pi_L(E)) > 0.$$

A few words about the proof

Want to show: E with $0 < \mathcal{H}^1(E) < \infty$ is purely unrectifiable iff

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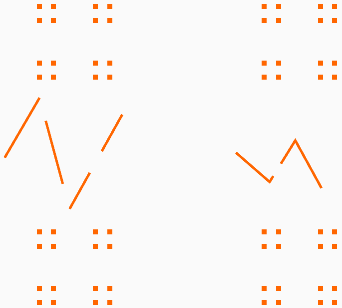

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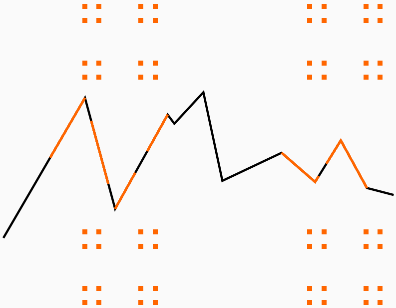

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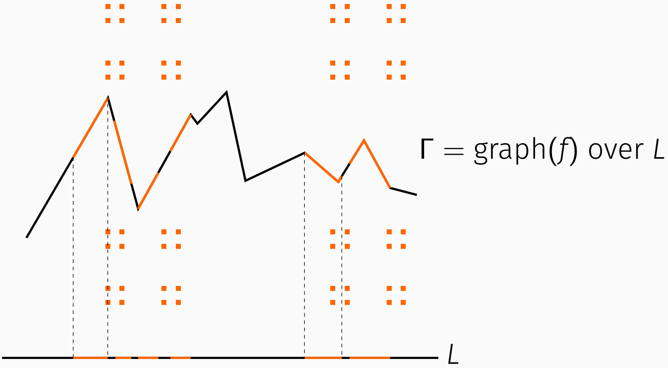

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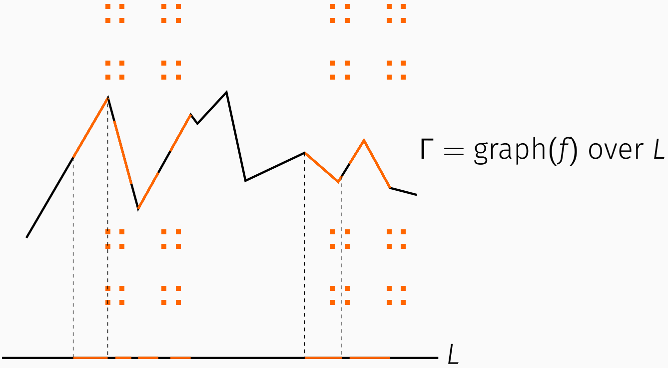

$\pi_L|_\Gamma$ is bilipschitz! $\quad \Rightarrow \quad \mathcal{H}^1(E \cap \Gamma) \approx \mathcal{H}^1(\pi_L(E \cap \Gamma))$

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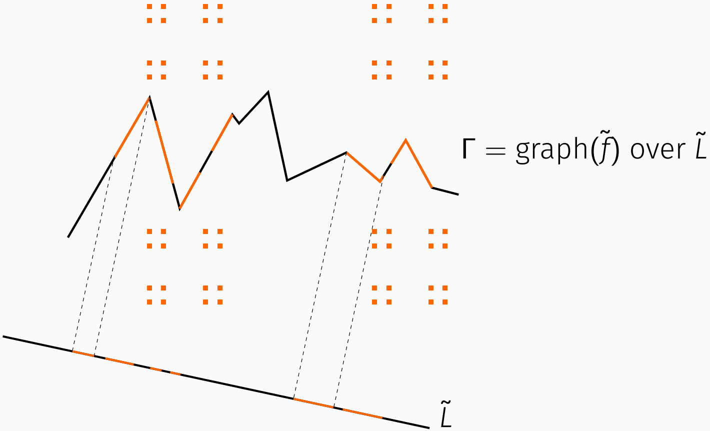

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- *Quantitatively* large projections in *quantitatively* many directions imply that E is *quantitatively* rectifiable (Orponen 2020)

Thank you!