On measures with L^2 bounded Riesz transform

To AD regularity and beyond

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The Riesz transform

Given $f \in L^2(\mathbb{R}^n)$ set

$$\mathcal{R}f(x) = \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mathcal{L}^n(y).$$

The Riesz transform

Given $f \in L^2(\mathbb{R}^n)$ and $\varepsilon > 0$ set

$$\mathcal{R}_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mathcal{L}^n(y).$$

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$$\mathcal{R}_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mathcal{L}^n(y).$$

Fact: the Riesz transform is bounded on $L^2(\mathbb{R}^n)$, in the sense that $\|\mathcal{R}_\varepsilon\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$ are bounded uniformly in ε.

The Riesz transform

Given a Radon measure μ on $\mathbb{R}^d$, $f \in L^2(\mu)$, and $\varepsilon > 0$ set

$$\mathcal{R}_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y).$$

We say that $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$ if $\|\mathcal{R}_{\mu,\varepsilon}\|_{L^2(\mu)\to L^2(\mu)}$ are bounded uniformly in ε.

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Question

What are the measures μ for which $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$?

Why do we care?

This question arises naturally in PDEs when studying

- the L^p solvability of the Dirichlet problem using the method of layer potentials,
- the removable sets for bounded analytic functions (in $\mathbb{R}^2$), or Lipschitz harmonic functions (in $\mathbb{R}^n$, $n \geq 2$).

Some easy examples

Examples of measures μ on $\mathbb{R}^d$ for which $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$:

- $\mu(B(x,r)) \leq Cr^s$ for some $s > n$,

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Lemma (David '91)

Suppose that $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$, and μ does not contain atoms. Then,

$$\mu(B(x,r)) \leq Cr^n.$$

Densities

For a Radon measure μ on $\mathbb{R}^d$, $x \in \mathbb{R}^d$ and $r > 0$ set

$$\theta_\mu(x, r) = \frac{\mu(B(x, r))}{r^n}.$$

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David's lemma: if $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$, then $\theta_\mu(x, r) \leq C$.

We will say that μ is **n-AD-regular** if for $x \in \operatorname{supp} \mu$, $0 < r < \operatorname{diam}(\operatorname{supp} \mu)$

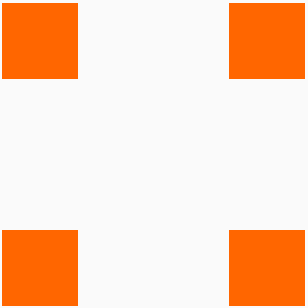
$$C^{-1} r^n \leq \mu(B(x, r)) \leq C r^n.$$

In other words,

$$\theta_\mu(x, r) \approx 1.$$

A negative example

The four-corner Cantor set $K \subset \mathbb{R}^2$ is an example of a set such that $\mu = \mathcal{H}^1\big|_K$ is 1-ADR but $\mathcal{R}_\mu$ is not bounded on $L^2(\mu)$.



K_1

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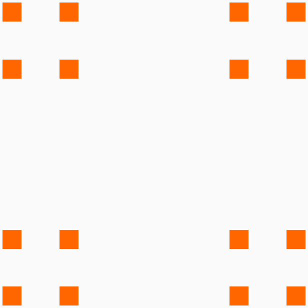

K_2

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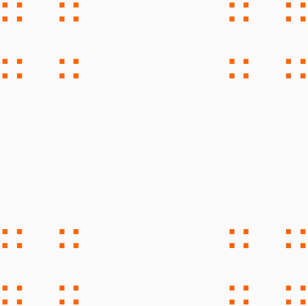

K_3

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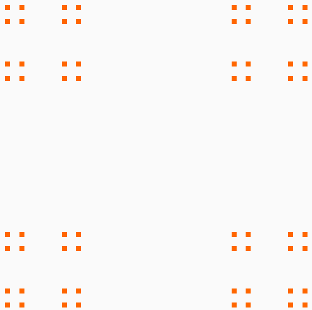
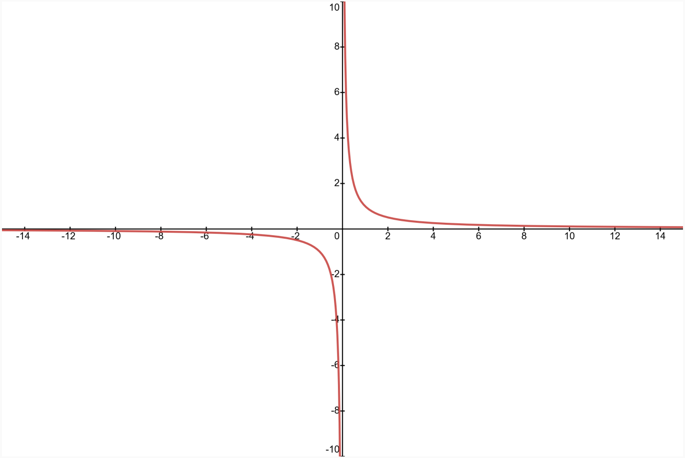


$$K = \bigcap_n K_n$$

Flatness is our friend

Recall that

$$\mathcal{R}_{\mu,\varepsilon}f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y).$$



What about Lipschitz graphs?

Let $\mu = \mathcal{H}^n\big|_{\Gamma}$. Then, $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$ provided that

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- Γ is a Lipschitz graph with sufficiently small Lipschitz constant (Calderón '77),
- Γ is a Lipschitz graph with an arbitrary Lipschitz constant (Coifman-McIntosh-Meyer '82),
- $n = 1$ and Γ is a 1-ADR curve (David '84).

Rectifiability

A set $E \subset \mathbb{R}^d$ is *n*-**rectifiable** if there exists a countable number of n-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^n\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

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Question

Suppose that E is an n-ADR, n-rectifiable set, and $\mu = \mathcal{H}^n\big|_E$. Does this imply that $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$?

Uniform rectifiability (David-Semmes ’91)

The answer is no. The notion of rectifiability is qualitative, while the boundedness of $\mathcal{R}_\mu$ is a quantitative property.

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We say that a measure μ is **uniformly n-rectifiable** if

- it is AD-regular
- there exists $L, \kappa > 0$ such that for all balls $B = B(x, r)$ centered at $\operatorname{supp} \mu$, $0 < r < \operatorname{diam}(\operatorname{supp} \mu)$, there exists a Lipschitz map $g : \mathbb{R}^n \to \mathbb{R}^d$, $\operatorname{Lip}(g) \leq L$, such that

$$\mu(B \cap g(B^n(0, r))) \geq \kappa \mu(B).$$

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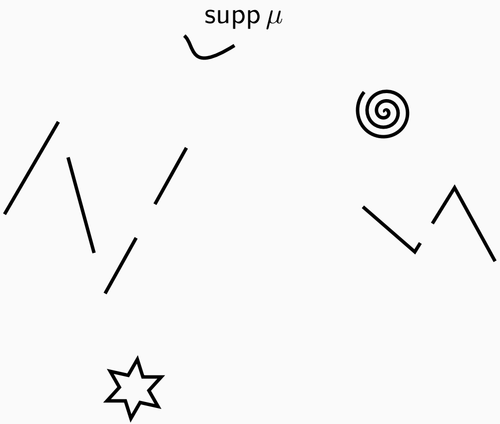

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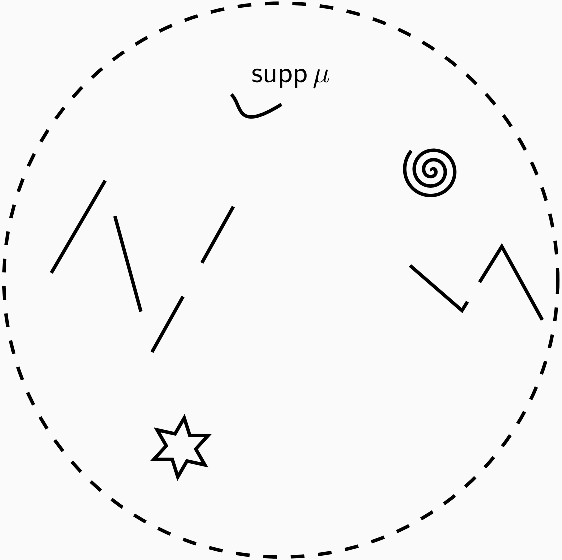

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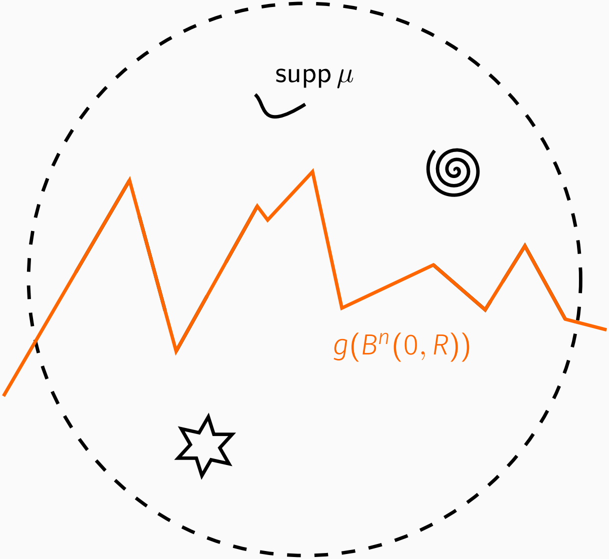

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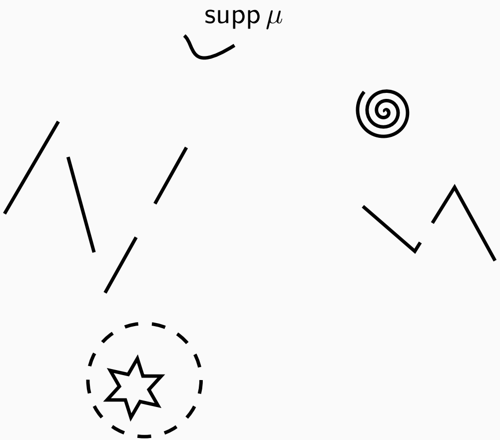

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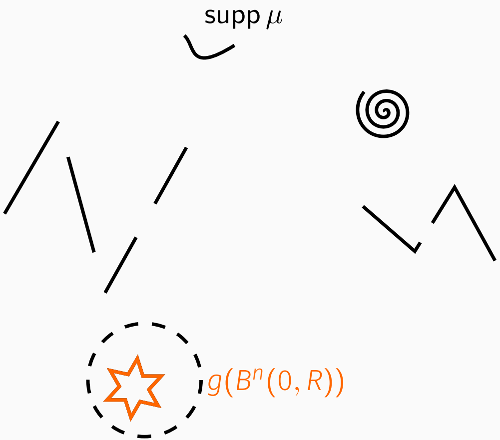

Uniform rectifiability and SIOs

Theorem (David-Semmes '91)

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

all "nice" SIOs are bounded on $L^2(\mu)$	$\Leftrightarrow$	μ is uniformly rectifiable.

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David-Semmes conjecture

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

$$\mathcal{R}_\mu \text{ is bounded on } L^2(\mu) \quad \Leftrightarrow \quad \mu \text{ is uniformly rectifiable.}$$

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David-Semmes conjecture

Suppose μ is n-AD-regular measure on $\mathbb{R}^d$. Then,

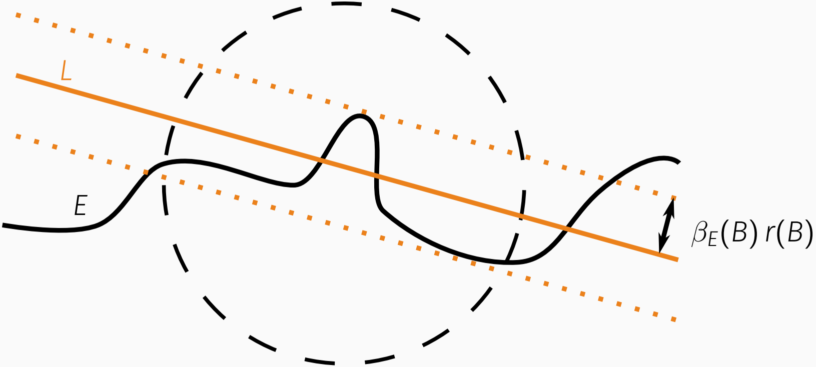
$$\mathcal{R}_\mu \text{ is bounded on } L^2(\mu) \quad \Leftrightarrow \quad \mu \text{ is uniformly rectifiable.}$$

True for $n = 1$ (Mattila-Melnikov-Verdera 1996) and $n = d - 1$ (Nazarov-Tolsa-Volberg 2012).

Beyond AD-regular measures

Given $E \subset \mathbb{R}^d$ and a ball B, $E \cap B \neq \varnothing$, the **$\beta$ number** of E at B is

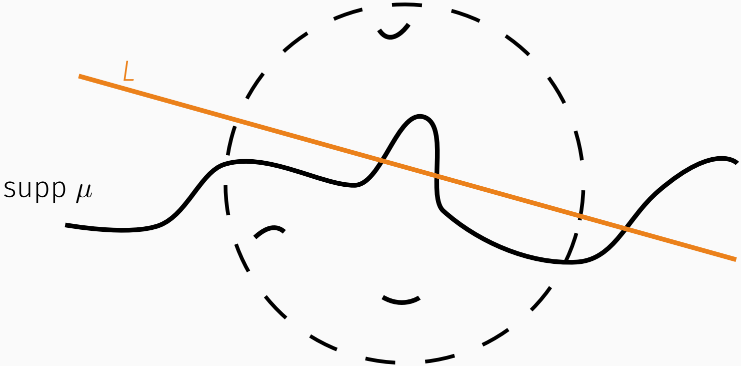
$$\beta_E(B) = \inf_{L} \sup_{x \in E \cap B} \frac{\operatorname{dist}(x, L)}{r(B)}.$$



β_2 numbers (David-Semmes '91)

Given a measure μ and a ball $B = B(x, r)$, the β_2 **number** of μ at B is

$$\beta_{\mu,2}(B) = \beta_{\mu,2}(x, r) = \inf_L \left(r(B)^{-n} \int_B \left(\frac{\operatorname{dist}(x, L)}{r(B)} \right)^2 d\mu(x) \right)^{1/2}.$$



β_2 numbers and uniform rectifiability

Theorem (David-Semmes '91)

Let μ be an n-ADR measure on $\mathbb{R}^d$. Then μ is uniformly n-rectifiable iff for all $z \in \operatorname{supp}\mu$, $R > 0$

$$\int_{B(z,R)} \int_0^R \beta_{\mu,2}(x,r)^2 \frac{dr}{r} d\mu(x) \leq CR^n.$$

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Corollary

Suppose that $n = 1$ or $n = d - 1$, and μ is an n-ADR measure on $\mathbb{R}^d$. Then, $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$ iff for all $z \in \operatorname{supp}\mu$, $R > 0$

$$\int_{B(z,R)} \int_0^R \beta_{\mu,2}(x,r)^2 \frac{dr}{r} d\mu(x) \leq CR^n.$$

Theorem (Azzam-Tolsa '15)

Suppose that $n = 1$ and μ is an atomless Radon measure on $\mathbb{R}^2$. Then, $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$ iff $\theta_\mu(x,r) \leq C$ and for all balls $B \subset \mathbb{R}^2$

$$\int_B \int_0^{r(B)} \beta_{\mu,2}(x,r)^2 \, \theta_\mu(x,r) \, \frac{dr}{r} d\mu(x) \leq C\mu(B).$$

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Theorem (Girela-Sarrión '19)

Suppose that μ is a Radon measure on $\mathbb{R}^d$. Assume that $\theta_\mu(x,r) \leq C$ and for all balls $B \subset \mathbb{R}^d$

$$\int_B \int_0^{r(B)} \beta_{\mu,2}(x,r)^2\, \theta_\mu(x,r)\, \frac{dr}{r} d\mu(x) \leq C\mu(B).$$

Then, all "nice" SIOs are bounded on $L^2(\mu)$.

Theorem (D.-Tolsa, Tolsa)

Suppose that μ is a Radon measure on $\mathbb{R}^{n+1}$ with $\theta_\mu(x,r) \leq C$. Assume that $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$. Then, for all balls $B \subset \mathbb{R}^{n+1}$

$$\int_B \int_0^{r(B)} \beta_{\mu,2}(x,r)^2\, \theta_\mu(x,r)\, \frac{dr}{r} d\mu(x) \leq C\mu(B).$$

New results

Theorem (D.-Tolsa, Tolsa)

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Corollary

Suppose that μ is an atomless Radon measure on $\mathbb{R}^{n+1}$. Then, $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$ iff $\theta_\mu(x,r) \leq C$ and for all balls $B \subset \mathbb{R}^{n+1}$

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Reduction to compactly supported measures

The proof reduces to showing the following:

Theorem

Suppose that μ is a compactly supported Radon measure on $\mathbb{R}^{n+1}$ with $\theta_\mu(x,r) \leq C$. Assume that "$\mathcal{R}\mu \in L^2(\mu)$." Then,

$$\iint_0^\infty \beta_{\mu,2}(x,r)^2\, \theta_\mu(x,r)\, \frac{dr}{r} d\mu(x) \lesssim \|\mu\| + \|\mathcal{R}\mu\|^2_{L^2(\mu)}.$$

Two papers?

Theorem (D.-Tolsa)

Suppose that μ is as before. Then,

$$\iint_0^\infty \beta_{\mu,2}(x,r)^2\,\theta_\mu(x,r)\,\frac{dr}{r}d\mu(x) \lesssim \|\mu\| + \|\mathcal{R}\mu\|^2_{L^2(\mu)} + \sum_{Q\in \mathrm{HE}} \mathcal{E}(4Q).$$

Theorem (Tolsa)

Suppose that μ is as before. Then,

$$\sum_{Q\in \mathrm{HE}} \mathcal{E}(4Q) \lesssim \|\mu\| + \|\mathcal{R}\mu\|^2_{L^2(\mu)}.$$

The proofs build up on techniques from [Eiderman-Nazarov-Volberg '14], [Nazarov-Tolsa-Volberg '14], [Reguera-Tolsa '16], [Jaye-Nazarov-Reguera-Tolsa '20]...

Some corollaries

Our results, together with [Azzam-Tolsa '15] and [Girela-Sarrión '19] give

Corollary 1

Suppose that μ is atomless, and that $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is bilipschitz. Set $\sigma = \varphi_{\#}\mu$. If $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$, then $\mathcal{R}_\sigma$ is bounded on $L^2(\sigma)$.

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Together with results from [Volberg ’03] we get also

Corollary 2

Suppose that $E \subset \mathbb{R}^{n+1}$, and that $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is bilipschitz. $\varphi(E)$ is removable for Lipschitz harmonic functions iff E is removable for Lipschitz harmonic functions.

Thank you!