Quantifying Besicovitch projection theorem

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Rectifiability

A set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of 1-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^1\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

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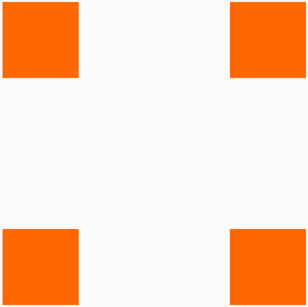
We say that $F \subset \mathbb{R}^2$ is purely unrectifiable if for every 1-dimensional Lipschitz graph Γ

$$\mathcal{H}^1(F \cap \Gamma) = 0.$$

Four-corner Cantor set



K_1

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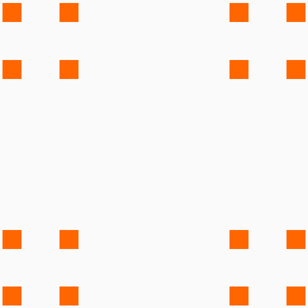

K_2

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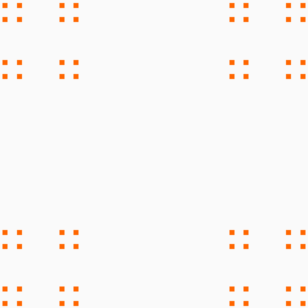

K_3

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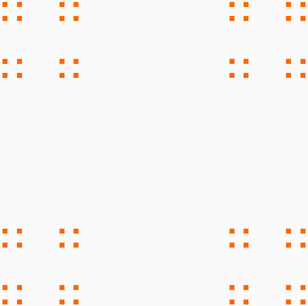

$$K = \bigcap_n K_n$$

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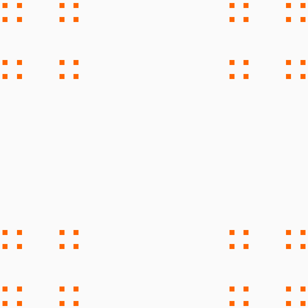

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Fact

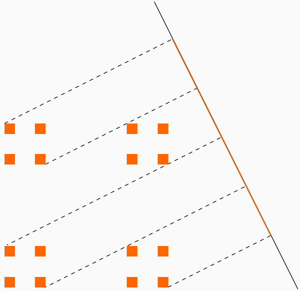
Any set E with $0 < \mathcal{H}^1(E) < \infty$ can be decomposed $E = R \cup U$ with R rectifiable and U purely unrectifiable.

Projections and rectifiability

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. Then, E is purely unrectifiable if and only if

$$\mathcal{H}^1(\pi_\theta(E)) = 0 \quad \text{for a.e. } \theta \in (0, \pi).$$



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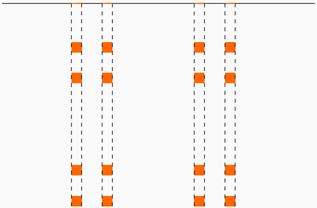

Possible extensions of Besicovitch's theorem

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- **generalized projections** (Peres, Schlag, Hovila, Järvenpää, Järvenpää, Ledrappier, Bond, Volberg, Cladek, Davey, Taylor)

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- **quantitative version** (Mattila, Tao, Łaba, Zhai, Bateman, Volberg, Bond, Nazarov, Wilson, Martikainen, Orponen, Bongers, Marshall)

Quantifying Besicovitch's theorem

Define **Favard length** of E as

$$\mathrm{Fav}(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

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Problem

Can we quantify the dependence of $\mathrm{Lip}(\Gamma)$ and $\mathcal{H}^1(E \cap \Gamma)$ on $\mathrm{Fav}(E)$?

Why is this interesting?

- fits into the framework of the quantitative rectifiability field, connections to PDEs and SIOs
- seems necessary for the solution of Vitushkin's conjecture

Naive conjecture...

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Naive conjecture

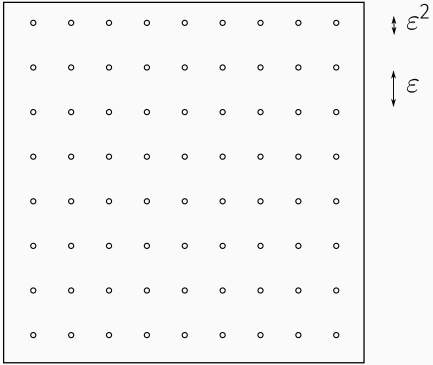
Let $E \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$. Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

... is false

For any $\varepsilon > 0$ there exists a set $E = E_\varepsilon \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathsf{Fav}(E) \gtrsim 1$ such that for all L-Lipschitz graphs Γ

$$\mathcal{H}^1(E \cap \Gamma) \lesssim L\varepsilon.$$



E consists of ε^{-2} uniformly distributed circles of radius ε^2.

Reasonable conjecture

We say that a set $E \subset \mathbb{R}^2$ is **AD-regular** if for any $x \in E$ and $0 < r < \operatorname{diam}(E)$ we have

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x, r)) \leq Cr.$$

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Previous results

Reasonable conjecture

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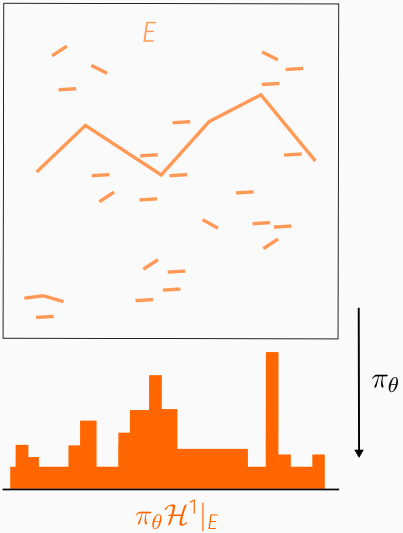
Significant progress towards the conjecture due to:

- Orponen 2021: sets with "plenty of big projections,"
- Martikainen-Orponen 2018: sets with projections in L^2.

Sets with projections in L^2

Big projections vs projections in L^p

Denote by $\pi_\theta \mathcal{H}^1|_E$ the pushforward of $\mathcal{H}^1|_E$ by π_θ.



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Observation

If $\pi_\theta \mathcal{H}^1|_E \in L^p$ for some $p > 1$, then

$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim \frac{\mathcal{H}^1(E)^{p'}}{\|\pi_\theta \mathcal{H}^1|_E\|_p^{p'}}.$$

Indeed:

$$\mathcal{H}^1(E) = \int_{\pi_\theta(E)} \pi_\theta \mathcal{H}^1|_E(x)\, dx \leq \left(\int \pi_\theta \mathcal{H}^1|_E(x)^p\, dx \right)^{1/p} \mathcal{H}^1(\pi_\theta(E))^{1/p'}.$$

Sets with projections in L^2

Theorem (Martikainen-Orponen 2018)

Let $E \subset [0,1]^2$ be an AD-regular set with $\mathcal{H}^1(E) \sim 1$. Suppose that there exists an arc $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that

$$\int_G \|\pi_\theta \mathcal{H}^1|_E\|_{L^2}^2 \, d\theta \lesssim 1.$$

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

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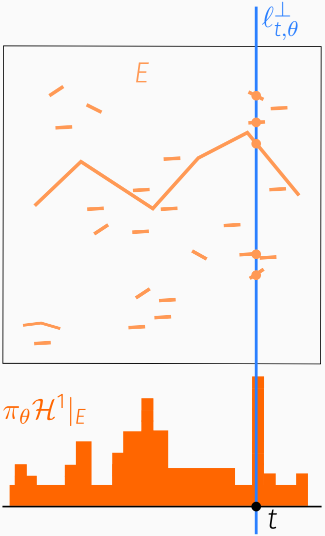
Projections in L^2 are special



$$\int_{\pi_\theta(E)} \pi_\theta \mathcal{H}^1|_E(t)^2 \, dt \sim \int_{\pi_\theta(E)} \#\{\ell_{t,\theta}^{\perp} \cap E\}^2 \, dt$$

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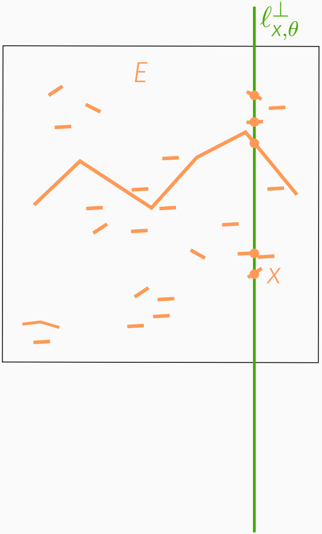

$$\int_{\pi_\theta(E)} \pi_\theta \mathcal{H}^1|_E(t)^2 \, dt \sim \int_{\pi_\theta(E)} \#\{\ell_{t,\theta}^{\perp} \cap E\}^2 \, dt = \int_E \#\{\ell_{x,\theta}^{\perp} \cap E\} \, dx.$$

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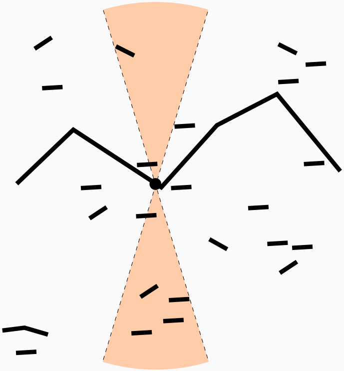

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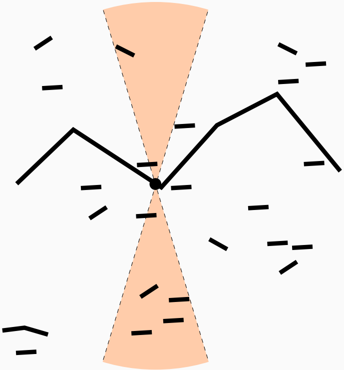
$$\sim \int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x,G,r))}{r} \frac{dr}{r} \, dx.$$



$$X(x,G) = \bigcup_{\theta \in G} \ell_{x,\theta}^\perp$$

$$X(x,G,r) = X(x,G) \cap B(x,r)$$

Projections in L^2 are special

In fact, one can use Fourier analysis to show:

Theorem (Chang-Tolsa 2020)

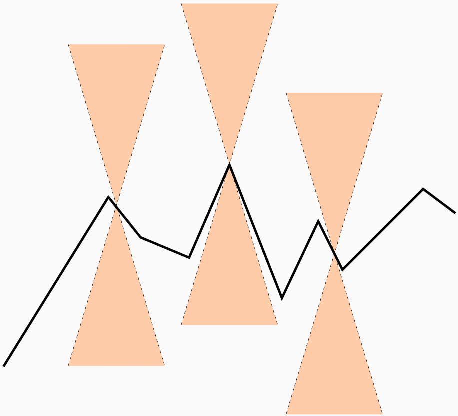
Let μ be a finite, compactly supported measure on $\mathbb{R}^2$, and $G \subset \mathbb{S}^1$ an open set. Then,

$$\int\int_0^\infty \frac{\mu(X(x,G,r))}{r} \frac{dr}{r}\, d\mu(x) \lesssim \int_G \|\pi_\theta \mu\|_{L^2}^2 \, d\theta.$$

Why is this useful?

Recall: $E \subset \mathbb{R}^2$ is a subset of a Lipschitz graph iff there exists an open cone X such that

$$x \in E \quad \Rightarrow \quad E \cap X(x) = \varnothing.$$



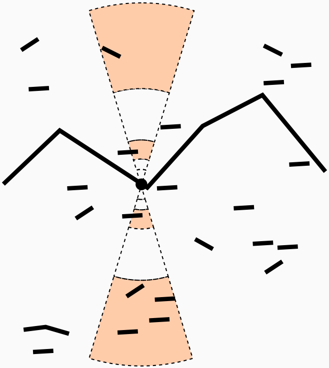
Why is this useful?

In our setting, the estimate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x,G,r))}{r} \frac{dr}{r}\, dx \lesssim 1$$

can be used to show that for most $x \in E$

$$\#\{j \in \mathbb{Z} \;:\; E \cap X(x,G,2^{-j-1},2^{-j}) \neq \varnothing\} \lesssim 1.$$



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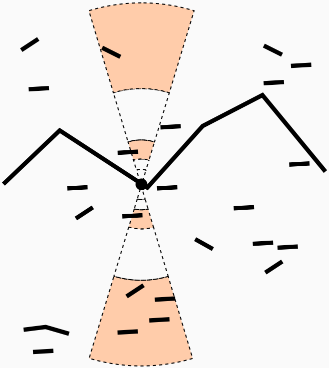

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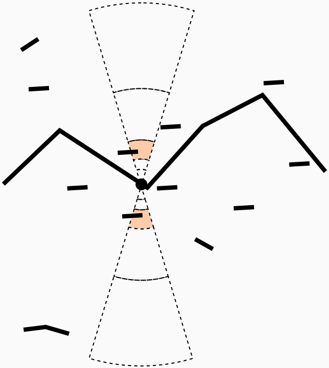

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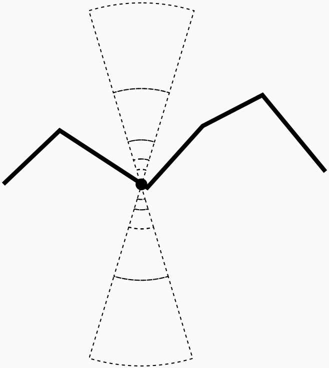

New result

Sets with projections in L^∞

Theorem (D. 2022)

Let $E \subset [0,1]^2$ be an AD-regular set with $\mathcal{H}^1(E) \sim 1$. Suppose that there exists a measurable $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that

$$\|\pi_\theta \mathcal{H}^1|_E\|_{L^\infty} \lesssim 1 \quad \text{for } \theta \in G.$$

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

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Why is this significant?

Reasonable conjecture

Let $E \subset [0,1]^2$ be an AD-regular set with $\mathcal{H}^1(E) \sim 1$ and $\mathbf{Fav}(E) \gtrsim 1$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

Note that $\mathbf{Fav}(E) \gtrsim 1$ if and only if there exists a measurable $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ such that

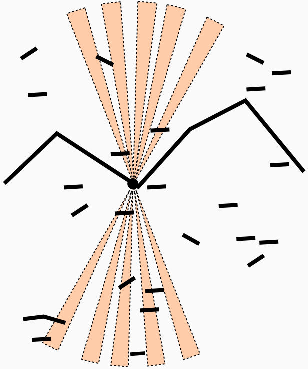
$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim 1 \quad \text{for } \theta \in G.$$

Difficulty 1. We can still get the estimate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G, r))}{r} \frac{dr}{r} \, dx \lesssim 1,$$

but now we cannot transform it to

$$\{j \in \mathbb{Z} \ : \ E \cap X(x, G, 2^{-j-1}, 2^{-j}) \neq \varnothing\} \lesssim 1.$$



New difficulties

Recall: $E \subset \mathbb{R}^2$ is a subset of a Lipschitz graph iff there exists an open cone X such that

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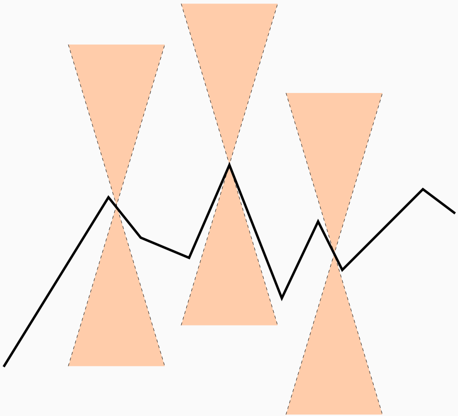
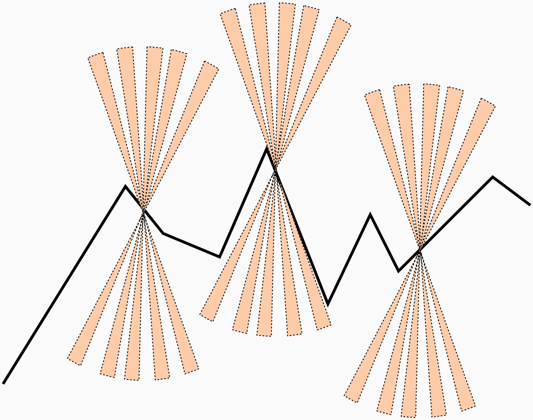


New difficulties

Difficulty 2. We are missing a characterization of Lipschitz graphs in terms of the “irregular, star-shaped” cones.



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Question

Suppose that $E \subset [0,1]^2$ is AD-regular with $\mathcal{H}^1(E) \sim 1$, and satisfies

$$x \in E \quad \Rightarrow \quad E \cap X(x, G) = \emptyset$$

for some $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$.

- Is E rectifiable?
- Is there a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1?$$

About the proof

Idea of the proof

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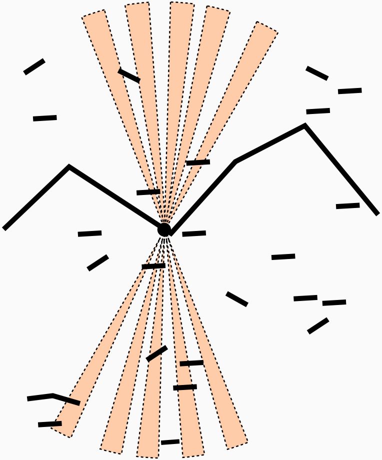
We prove that there exists an arc $J \subset \mathbb{S}^1$ with $\mathcal{H}^1(J) \sim 1$ such that

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, J, r))}{r} \frac{dr}{r} \, dx \lesssim 1.$$

Then, we can use the result of Martikainen-Orponen to find the desired big piece of a Lipschitz graph.

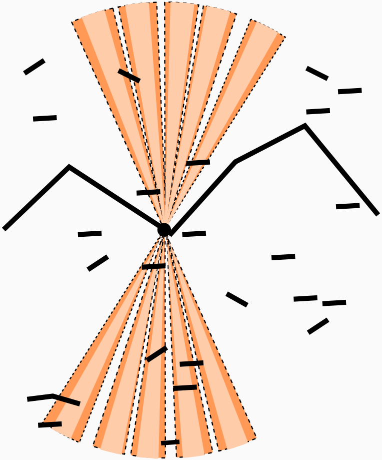
Good directions propagate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G, r))}{r} \frac{dr}{r}\, dx \lesssim 1.$$



Good directions propagate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G', r))}{r} \frac{dr}{r}\, dx \lesssim \int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G, r))}{r} \frac{dr}{r}\, dx \lesssim 1.$$



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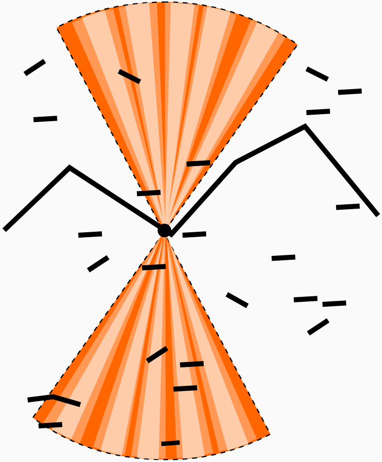

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Main propositon

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Suppose that

- $E \subset [0,1]^2$ is an AD-regular set with $\mathcal{H}^1(E) \sim 1$,
- $J \subset \mathbb{S}^1$ is an arc, and $G_J \subset J$ is measurable with $\mathcal{H}^1(J \setminus G_J) \leq \varepsilon \mathcal{H}^1(J)$,

Good directions propagate

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- technical assumptions involving $\|\pi_\theta \mathcal{H}^1|_E\|_\infty$.

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- technical assumptions involving $\|\pi_\theta \mathcal{H}^1|_E\|_\infty$.

Then,

$$\int_E \int_0^1 \frac{\mathcal{H}^1(E \cap X(x, 3J, r))}{r} \frac{dr}{r} d\mathcal{H}^1(x) \lesssim \int_E \int_0^1 \frac{\mathcal{H}^1(E \cap X(x, G_J, r))}{r} \frac{dr}{r} d\mathcal{H}^1(x) + \mathcal{H}^1(J).$$

Questions

Can we relax the L^∞-assumption to the L^2-assumptions?

Question 1

Let $E \subset [0,1]^2$ be an AD-regular set with $\mathcal{H}^1(E) \sim 1$. Suppose that there exists a measurable $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that

$$\|\pi_\theta \mathcal{H}^1|_E\|_{L^2} \lesssim 1 \quad \text{for } \theta \in G.$$

Does there exist a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1?$$

Questions

Can a similar approach be used to prove the Reasonable conjecture?

Question 2

Suppose that

- $E \subset [0,1]^2$ is an AD-regular set with $\mathcal{H}^1(E) \sim 1$,
- $J \subset \mathbb{S}^1$ is an arc, and $G_J \subset J$ is measurable with $\mathcal{H}^1(J \setminus G_J) \leq \varepsilon \mathcal{H}^1(J)$,
- we have

$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim 1 \quad \text{for } \theta \in G_J$$

Then, do we have

$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim 1 \quad \text{for } \theta \in 3J?$$

Thank you!