From orthogonal projections to Furstenberg sets

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Based on joint work with Tuomas Orponen and Michele Villa

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Projections and dimension

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Theorem (Marstrand 1954)

If $E \subset \mathbb{R}^2$ has $\dim_{\mathrm{H}}(E) \leq 1$, then for a.e. $\theta \in \mathbb{S}^1$

$$\dim_{\mathrm{H}}(\pi_\theta(E)) = \dim_{\mathrm{H}}(E).$$

If $\dim_{\mathrm{H}}(E) > 1$, then for a.e. $\theta \in \mathbb{S}^1$

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Theorem (Kaufman 1968)

If $\dim_{\mathrm{H}}(E) > 1$, then E supports a probability measure μ with

$$\int_{\mathbb{S}^1} \|\pi_\theta \mu\|_{L^2}^2 \, d\theta < \infty.$$

Projections of Frostman measures

Recall: a Radon measure μ on $\mathbb{R}^2$ is s-Frostman, $\mu \in \mathcal{M}_s$, if

$$\mu(B(x,r)) \leq r^s.$$

Frostman's lemma:

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Question

Let $1 < s \leq 2$. For which values of $1 \leq p, q \leq \infty$ does it hold that for all $\mu \in \mathcal{M}_s([0,1]^2)$ we have

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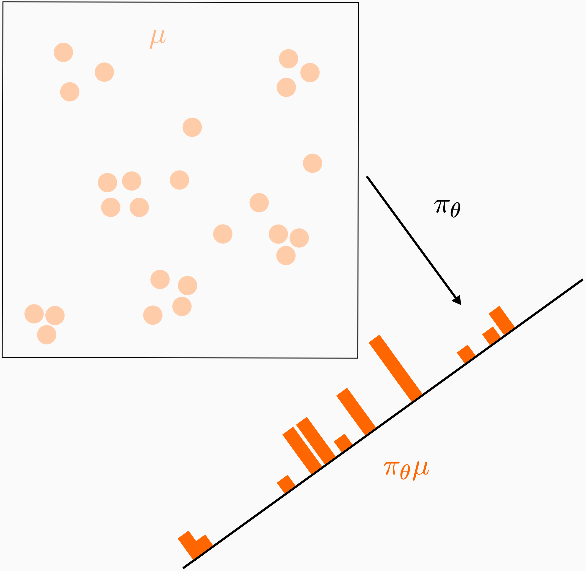

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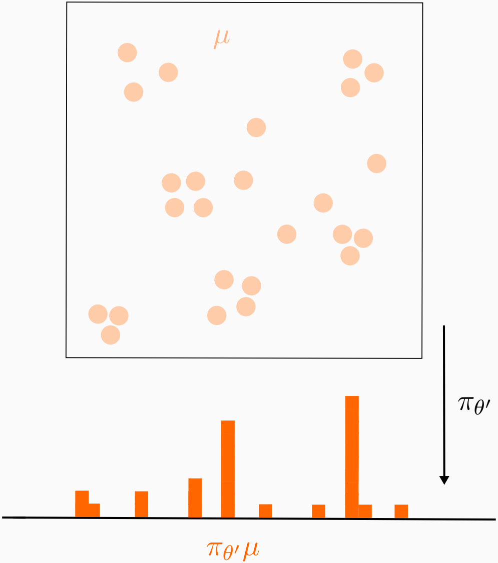

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- A trivial bound for $s = 2$:

$$\|\pi_\theta \mu\|_\infty \in L^\infty(\mathbb{S}^1) \qquad \forall \mu \in \mathcal{M}_2([0,1]^2).$$

Our result

Theorem (D.-Orponen-Villa)

Let μ be an s-Frostman measure on $\mathbb{R}^2$ with $1 < s \leq 2$. Then

$$\int_{\mathbb{S}^1} \|\pi_\theta \mu\|_{L^p}^p \, d\theta < \infty$$

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The proof:

- the desired estimate can be reinterpreted as an L^p-bound for a family of operators involving the X-ray transform and fractional Laplacian,
- we use Stein's complex interpolation theorem to interpolate between the L^2-estimate of Kaufman and the trivial L^∞ bound for $\mathcal{M}_2$.

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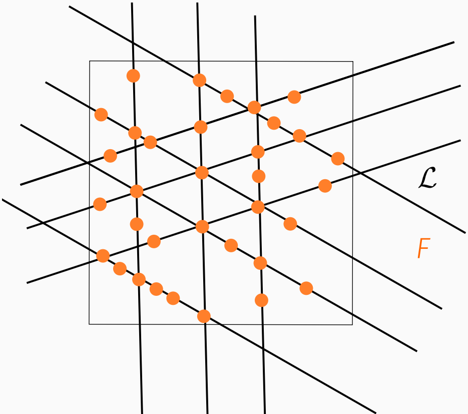
Remarks:

- The result is sharp.
- We also prove a higher-dimensional analogue.

Let $0 < s \leq 1$ and $0 < t \leq 2$. We say that a set $F \subset \mathbb{R}^2$ is an **(s,t)-Furstenberg set** if there exists a t-dimensional family of lines $\mathcal{L}$ such that for every $\ell \in \mathcal{L}$ we have

$$\dim_{\mathrm{H}}(F \cap \ell) \geq s.$$



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Why is this interesting?

- proposed and first studied by Wolff!
- fractal counterpart of the Kakeya conjecture
- connection to geometric combinatorics

What is known?

Previous results due to Wolff, Katz, Tao, Bourgain, Molter, Rela, Héra, Máthé, Keleti, Lutz, Stull, Shmerkin, Yavicoli, Orponen, Di Benedetto, Zahl...

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$$\dim_H(F_{s,t}) \geq \begin{cases} s+t & \text{for } s \in (0,1] \text{ and } t \in (0,s], \\ & \\ & \end{cases}$$

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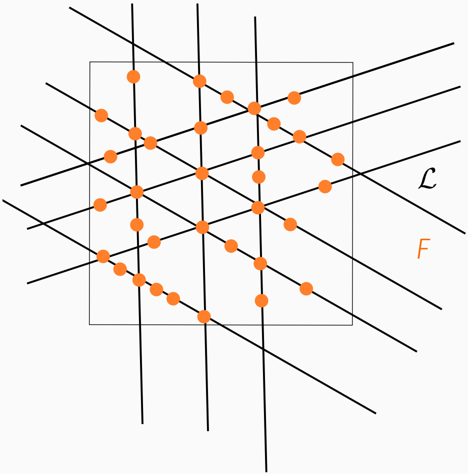
Theorem (D.-Orponen-Villa)

If $s \in (0,1]$ and $t \in (1,2]$, then

$$\dim_H(F_{s,t}) \geq 2s + (1-s)(t-1).$$

Proof idea

If $s \in (0,1]$ and $t \in (1,2]$, then $\dim_{\mathrm{H}}(F) \geq 2s + (1-s)(t-1)$.



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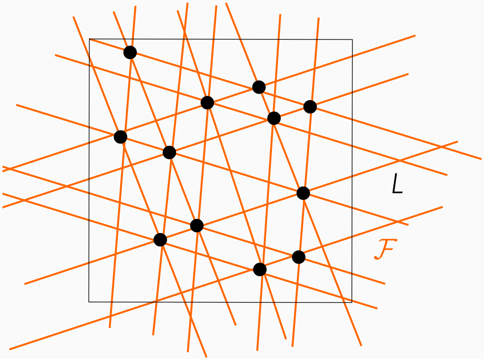

Step 1.

Use duality to obtain a t-dimensional set L and a collection of lines $\mathcal{F}$ such that for every $x \in L$

$$\dim_H(\mathcal{F}(x)) \geq s.$$

Let μ be a t-Frostman measure on L.

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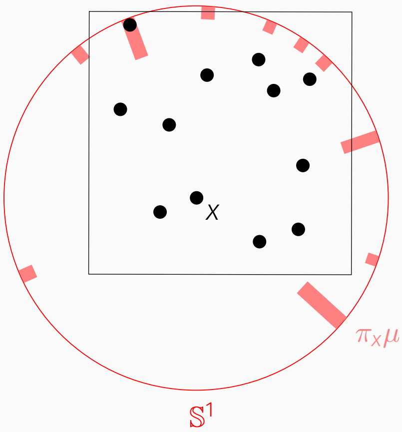

Step 2.

For $x \in L$ consider the "radial projections" of μ

$$\pi_x\mu(\Theta) = \mu\Big(\bigcup_{\theta\in\Theta} \ell_{x,\theta}\Big).$$

If the collection $\mathcal{F}$ is "small" then for many x the measures $\pi_x\mu$ will be quite singular:

$$\int \|\pi_x\mu\|^p_{L^p(\mathbb{S}^1)} d\mu(x)$$

$$\geq \dots(p, t, s, \dim_H(F))$$

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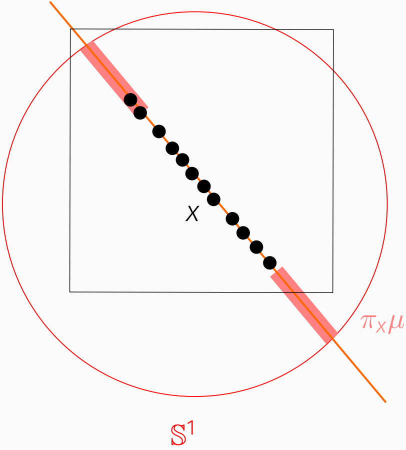

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Use Orponen's formula

$$\int \|\pi_x\mu\|_{L^p(\mathbb{S}^1)}^p d\mu(x) = \int_{\mathbb{S}^1} \|\pi_\theta\mu\|_{L^{p+1}}^{p+1} d\theta$$

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Step 4.

Use our result on projections of t-Frostman measures to get

$$\ldots(p,t,s,\dim_{\mathrm{H}}(F)) \leq \int \|\pi_x\mu\|^p_{L^p(\mathbb{S}^1)} d\mu(x) = \int_{\mathbb{S}^1} \|\pi_\theta\mu\|^{p+1}_{L^{p+1}} \, d\theta < \infty.$$

$\blacksquare$

Open problems

- What about other pairs of p and q in the projections problem? Particularly interesting the case $q = 1$: what's the largest $p = p(t)$ so that for all $\mu \in \mathcal{M}_t$

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- Our sharpness example only works for measures $\mu \in \mathcal{M}_s(\mathbb{R}^d)$ with $d - 1 < s \leq d$. Is the higher dimensional version of our projection result sharp for measures $\mu \in \mathcal{M}_s(\mathbb{R}^d)$ with $s < d - 1$?

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- Can our methods be extended to study (s, t)-Furstenberg sets for $t \leq 1$?

Thank you!