Vitushkin's conjecture and sets with plenty of big projections

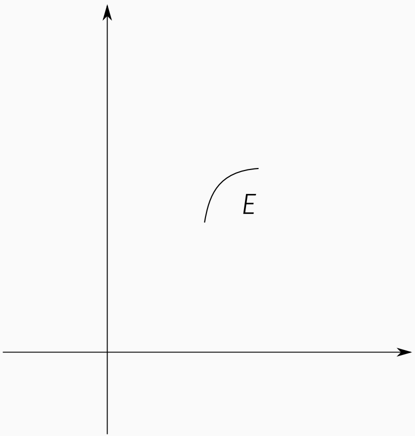
Damian Dąbrowski

Based on joint work with Michele Villa

JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

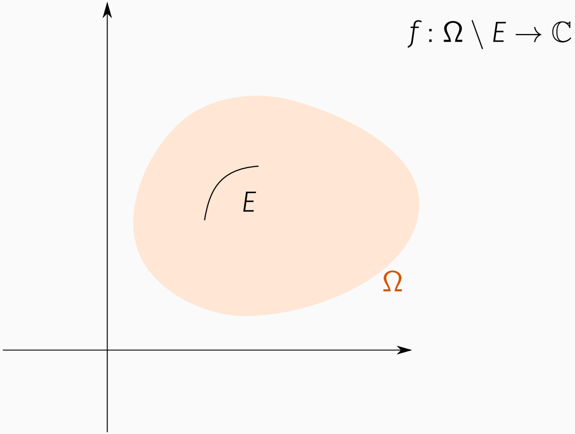
Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



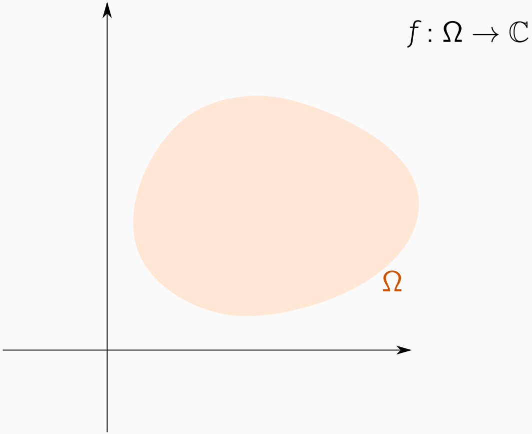
Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



In 1947 Ahlfors characterized removability in terms of **analytic capacity**:

$$E \text{ is removable} \quad \Leftrightarrow \quad \gamma(E) = 0,$$

where

$$\gamma(E) = \sup\{|f'(\infty)| \;:\; f : \mathbb{C} \setminus E \to \mathbb{C} \text{ analytic}, \; \|f\|_\infty \leq 1\},$$
$$f'(\infty) = \lim_{z\to\infty} z(f(z) - f(\infty)).$$

Painlevé Problem

Painlevé Problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Painlevé Problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Facts:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.

Painlevé Problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Facts:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.
- If E is a segment, then $\gamma(E) = c\,\mathcal{H}^1(E)$.

Painlevé Problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Facts:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.
- If E is a segment, then $\gamma(E) = c\,\mathcal{H}^1(E)$.

Question

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0?$$

Painlevé Problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

Facts:

- If $\mathcal{H}^1(E) = 0$, then $\gamma(E) = 0$.
- If $\dim_H(E) > 1$, then $\gamma(E) > 0$.
- If E is a segment, then $\gamma(E) = c\,\mathcal{H}^1(E)$.

Question

$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$? No!

There are sets $E \subset \mathbb{C}$ with $\gamma(E) = 0$ and $0 < \mathcal{H}^1(E) < \infty$.
(Vitushkin 1959, Garnett, Ivanov 1970s)

Vitushkin's conjecture

The sets constructed by Vitushkin, Garnett and Ivanov had very small projections. More precisely, they satisfied

$$\mathcal{H}^1(\pi_\theta(E)) = 0$$

for a.e. direction $\theta \in [0, \pi]$.

Vitushkin's conjecture

The sets constructed by Vitushkin, Garnett and Ivanov had very small projections. More precisely, they satisfied

$$\mathcal{H}^1(\pi_\theta(E)) = 0$$

for a.e. direction $\theta \in [0, \pi]$.

Define **Favard length** of E as

$$\mathrm{Fav}(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0$$

Rectifiability

A set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of 1-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^1\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

Rectifiability

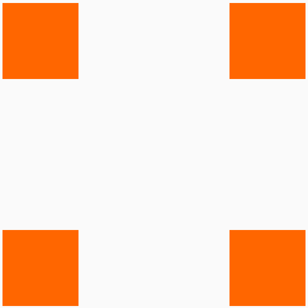
A set $E \subset \mathbb{R}^2$ is rectifiable if there exists a countable number of 1-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^1\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

We say that $F \subset \mathbb{R}^2$ is purely unrectifiable if for every 1-dimensional Lipschitz graph Γ

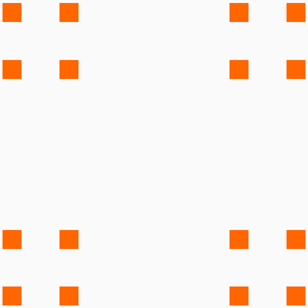
$$\mathcal{H}^1(F \cap \Gamma) = 0.$$

Four-corner Cantor set



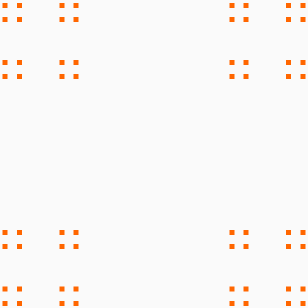
K_1

Four-corner Cantor set



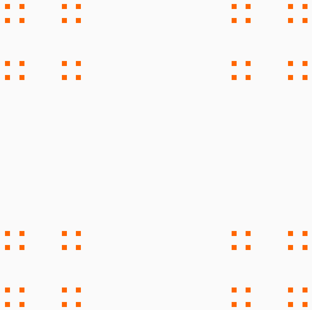
K_2

Four-corner Cantor set



K_3

Four-corner Cantor set



$$K = \bigcap_n K_n$$

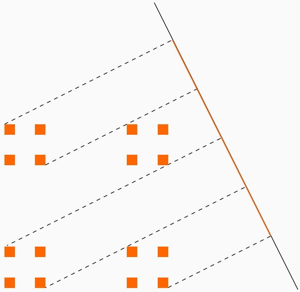
- an elementary argument gives $0 < \mathcal{H}^1(K) < \infty$ and $\text{Fav}(K) = 0$,
- $\gamma(K) = 0$ (Garnett, Ivanov 70s)

Projections and rectifiability

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. Then

$$E \text{ is purely unrectifiable} \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0.$$

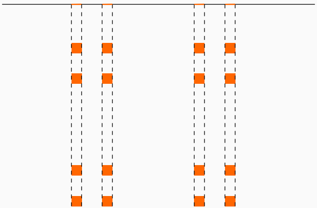


Projections and rectifiability

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. Then

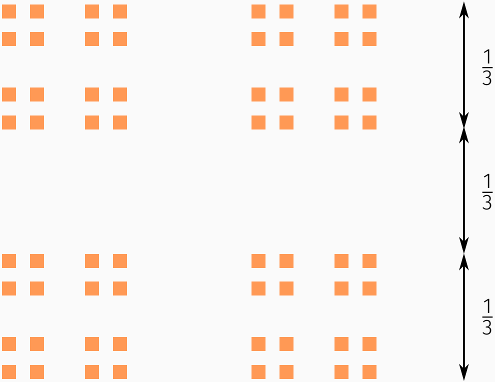
$$E \text{ is purely unrectifiable} \quad \Leftrightarrow \quad \mathrm{Fav}(E) = 0.$$



Beware of sets with $\mathcal{H}^1(E) = \infty$!

Besicovitch projection theorem fails terribly for sets with $\mathcal{H}^1(E) = \infty$: there exist purely unrectifiable sets E with $\mathsf{Fav}(E) > 0$.

Example: $E = C \times C$, where C is the middle-thirds Cantor set.



Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathsf{Fav}(E) = 0$$

In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)

Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad \text{Fav}(E) = 0$$

In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)

In the case $\mathcal{H}^1(E) = \infty$, Vitushkin's conjecture is **false** (Mattila '86).

Open problems

Theorem (Jones-Murai '88)

There exists a set with $\mathsf{Fav}(E) = 0$ and $\gamma(E) > 0$.

Open problems

Theorem (Jones-Murai '88)

There exists a set with $\mathrm{Fav}(E) = 0$ and $\gamma(E) > 0$.

Problem 1 (qualitative)

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Open for sets $E \subset \mathbb{C}$ with $\dim_H(E) = 1$ and non-σ-finite $\mathcal{H}^1$-measure.

Open problems

Theorem (Jones-Murai '88)

There exists a set with $\mathrm{Fav}(E) = 0$ and $\gamma(E) > 0$.

Problem 1 (qualitative)

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Open for sets $E \subset \mathbb{C}$ with $\dim_H(E) = 1$ and non-σ-finite $\mathcal{H}^1$-measure.

Problem 2 (quantitative)

$$\gamma(E) \gtrsim \mathrm{Fav}(E)?$$

$$\gamma(E) \gtrsim_{\mathrm{Fav}(E)} 1?$$

Open even in the case $\mathcal{H}^1(E) < \infty$.

What happens in the case $\mathcal{H}^1(E) < \infty$?

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph Γ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

What happens in the case $\mathcal{H}^1(E) < \infty$?

$$\mathrm{Fav}(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph Γ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, we get

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón '77)}}{>} 0.$$

Two big problems

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem we get

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón '77)}}{>} 0.$$

Problems:

Two big problems

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem we get

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón '77)}}{>} 0.$$

Problems:

1. The Besicovitch projection theorem **fails** for sets with $\mathcal{H}^1(E) = \infty$!

Two big problems

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem we get

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón '77)}}{>} 0.$$

Problems:

1. The Besicovitch projection theorem **fails** for sets with $\mathcal{H}^1(E) = \infty$!
2. There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

$$\gamma(E \cap \Gamma) \gtrsim_L \mathcal{H}^1(E \cap \Gamma)...$$

Two big problems

If $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then by the Besicovitch projection theorem we get

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón '77)}}{>} 0.$$

Problems:

1. The Besicovitch projection theorem **fails** for sets with $\mathcal{H}^1(E) = \infty$!
2. There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

$$\gamma(E \cap \Gamma) \gtrsim_L \mathcal{H}^1(E \cap \Gamma)\ldots$$

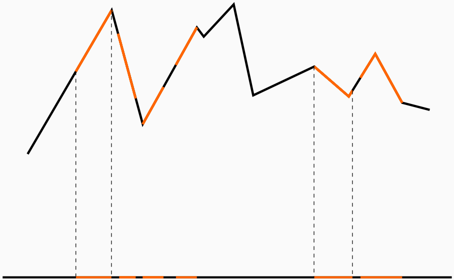
   ...but the Besicovitch projection theorem gives **no quantitative bound** neither on $\mathcal{H}^1(E \cap \Gamma)$, nor on $\mathrm{Lip}(\Gamma)$!

New results

Sets with plenty of big projections

We say that a set $E \subset \mathbb{R}^2$ has **plenty of big projections** (PBP) if there exists $\delta > 0$ such that for every $x \in E$ and $0 < r < \operatorname{diam}(E)$ we have a direction $\theta_{x,r} \in [0, \pi)$ such that

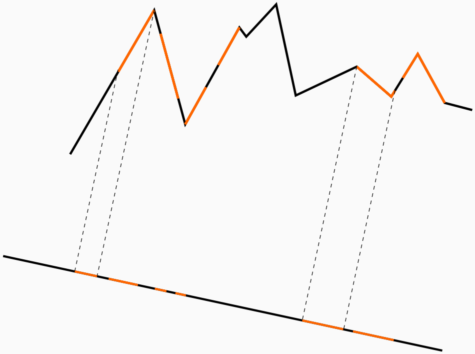
$$\mathcal{H}^1(\pi_\theta(E \cap B(x, r))) \geq \delta r \quad \text{for all} \quad |\theta - \theta_{x,r}| < \delta.$$



Sets with plenty of big projections

We say that a set $E \subset \mathbb{R}^2$ has plenty of big projections (PBP) if there exists $\delta > 0$ such that for every $x \in E$ and $0 < r < \mathrm{diam}(E)$ we have a direction $\theta_{x,r} \in [0, \pi)$ such that

$$\mathcal{H}^1(\pi_\theta(E \cap B(x, r))) \geq \delta r \quad \text{for all} \quad |\theta - \theta_{x,r}| < \delta.$$



Sets with plenty of big projections

We say that a set $E \subset \mathbb{R}^2$ has plenty of big projections (PBP) if there exists $\delta > 0$ such that for every $x \in E$ and $0 < r < \text{diam}(E)$ we have a direction $\theta_{x,r} \in [0, \pi)$ such that

$$\mathcal{H}^1(\pi_\theta(E \cap B(x,r))) \geq \delta r \quad \text{for all} \quad |\theta - \theta_{x,r}| < \delta.$$

Theorem (D.-Villa)

If a compact set $E \subset \mathbb{R}^2$ has PBP, then

$$\gamma(E) \gtrsim_\delta \text{diam}(E).$$

An analogous result holds in higher dimension and codimension for capacities $\Gamma_{d,n}(E)$.

Two crucial blackboxes

Geometric characterization of γ(E)

For a set $E \subset \mathbb{R}^2$ let $\mathcal{F}(E)$ be the set of Radon measures satisfying:

- $\operatorname{supp} \mu \subset E$,

Geometric characterization of γ(E)

For a set $E \subset \mathbb{R}^2$ let $\mathcal{F}(E)$ be the set of Radon measures satisfying:

- $\operatorname{supp} \mu \subset E$,
- $\mu(B(x,r)) \leq r$,

Geometric characterization of γ(E)

For a set $E \subset \mathbb{R}^2$ let $\mathcal{F}(E)$ be the set of Radon measures satisfying:

- $\operatorname{supp} \mu \subset E$,
- $\mu(B(x, r)) \leq r$,
- a flatness condition

$$\iint \beta_{\mu,2}(x, r)^2 \, \frac{\mu(B(x, r))}{r} \, \frac{dr}{r} d\mu(x) \leq \mu(E).$$

Geometric characterization of γ(E)

For a set $E \subset \mathbb{R}^2$ let $\mathcal{F}(E)$ be the set of Radon measures satisfying:

- $\operatorname{supp} \mu \subset E$,
- $\mu(B(x,r)) \leq r$,
- a flatness condition
$$\iint \beta_{\mu,2}(x,r)^2 \, \frac{\mu(B(x,r))}{r} \, \frac{dr}{r} d\mu(x) \leq \mu(E).$$

Examples:

- $\mathcal{L}^2|_{[0,1]^2} \in \mathcal{F}([0,1]^2)$
- $\mathcal{H}^1|_\Gamma \in \mathcal{F}(\Gamma)$ for any 1-Lipschitz graph Γ
- $\mathcal{F}(K) = \{0\}$ for the 4-corner Cantor set K

Geometric characterization of $\gamma(E)$

For a set $E \subset \mathbb{R}^2$ let $\mathcal{F}(E)$ be the set of Radon measures satisfying:

- $\operatorname{supp} \mu \subset E$,
- $\mu(B(x,r)) \leq r$,
- a flatness condition

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \leq \mu(E).$$

Theorem (Tolsa 2005, Azzam-Tolsa 2015)

$$\gamma(E) \sim \sup\{\mu(E) \ : \ \mu \in \mathcal{F}(E)\}$$

Takeaway: to show $\gamma(E) > 0$, it suffices to find a non-zero $\mu \in \mathcal{F}(E)$.

Big pieces of Lipschitz graphs

We say that a set $E \subset \mathbb{R}^2$ is **AD-regular** if for any $x \in E$ and $0 < r < \text{diam}(E)$ we have

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x, r)) \leq Cr.$$

Big pieces of Lipschitz graphs

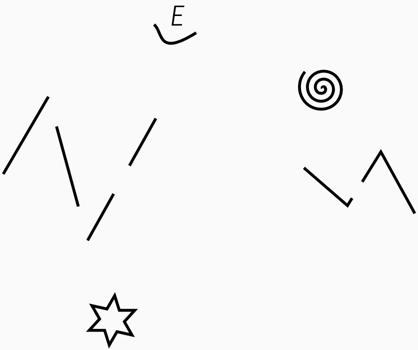
We say that a set $E \subset \mathbb{R}^2$ is **AD-regular** if for any $x \in E$ and $0 < r < \text{diam}(E)$ we have

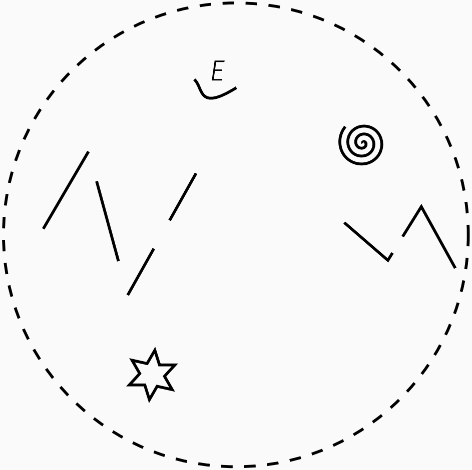
$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x,r)) \leq Cr.$$

An AD-regular set E contains **big pieces of Lipschitz graphs (BPLG)** if there exist $C, L > 0$ such that for every $x \in E$ and every $0 < r < \text{diam}(E)$ there exists an L-Lipschitz graph $\Gamma = \Gamma_{x,r}$ with

$$\mathcal{H}^1(E \cap \Gamma \cap B(x,r)) \geq Cr.$$

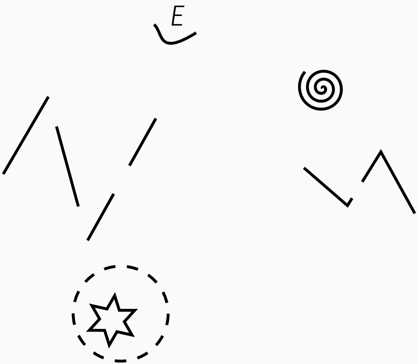
Big pieces of Lipschitz graphs



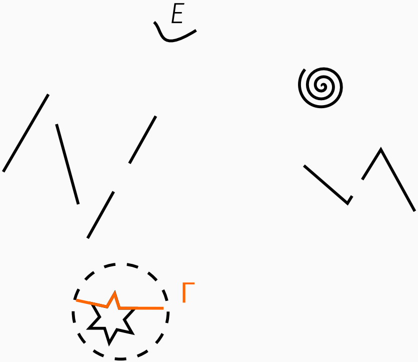

E

Big pieces of Lipschitz graphs

Big pieces of Lipschitz graphs



Big pieces of Lipschitz graphs



Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph Γ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathsf{Fav}(E) > 0$, then there exists a Lipschitz graph Γ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

Theorem (Orponen 2021)

Let E be AD-regular. If E has PBP, then it contains big pieces of Lipschitz graphs.

PBP and quantitative rectifiability

Theorem (David-Semmes 1991)

If E is AD-regular and has BPLG, then $\mu = \mathcal{H}^1|_E$ satisfies

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \lesssim \mu(E),$$

i.e. $\mu \in \mathcal{F}(E)$.

Corollary

If E is AD-regular and has PBP, then $\mu = \mathcal{H}^1|_E$ satisfies

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \lesssim \mu(E).$$

The proof

Sketch of the proof

Theorem (D.-Villa)

If a compact set $E \subset \mathbb{R}^2$ has PBP, then

$$\gamma(E) \gtrsim_\delta \operatorname{diam}(E).$$

Plan:

1. Take the usual Frostman measure μ on E. We'll show $\mu \in \mathcal{F}(E)$.

Sketch of the proof

Theorem (D.-Villa)

If a compact set $E \subset \mathbb{R}^2$ has PBP, then

$$\gamma(E) \gtrsim_\delta \mathrm{diam}(E).$$

Plan:

1. Take the usual Frostman measure μ on E. We'll show $\mu \in \mathcal{F}(E)$.
2. Multi-scale approximation of μ by a family of AD-regular measures $\{\eta_R\}_{R \in \mathrm{Roots}}$ (corona decomposition).

Sketch of the proof

Theorem (D.-Villa)

If a compact set $E \subset \mathbb{R}^2$ has PBP, then

$$\gamma(E) \gtrsim_\delta \operatorname{diam}(E).$$

Plan:

1. Take the usual Frostman measure μ on E. We'll show $\mu \in \mathcal{F}(E)$.
2. Multi-scale approximation of μ by a family of AD-regular measures $\{\eta_R\}_{R \in \text{Roots}}$ (corona decomposition).
3. Each η_R has PBP; apply Orponen's result to get an estimate on $\beta_{\eta_R,2}$.

Sketch of the proof

Theorem (D.-Villa)

If a compact set $E \subset \mathbb{R}^2$ has PBP, then

$$\gamma(E) \gtrsim_\delta \operatorname{diam}(E).$$

Plan:

1. Take the usual Frostman measure μ on E. We'll show $\mu \in \mathcal{F}(E)$.
2. Multi-scale approximation of μ by a family of AD-regular measures $\{\eta_R\}_{R \in \text{Roots}}$ (corona decomposition).
3. Each η_R has PBP; apply Orponen's result to get an estimate on $\beta_{\eta_R,2}$.
4. Transfer the estimates to $\beta_{\mu,2}$ to conclude $\mu \in \mathcal{F}(E)$.

Step 1. Frostman measure

Let $E \subset \mathbb{R}^2$ be a compact set with PBP. Our goal is to find a non-zero measure $\mu \in \mathcal{F}(E)$, i.e. a measure μ supported on E satisfying $\mu(B(x,r)) \leq r$ and

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \leq \mu(E).$$

Step 1. Frostman measure

Let $E \subset \mathbb{R}^2$ be a compact set with PBP. Our goal is to find a non-zero measure $\mu \in \mathcal{F}(E)$, i.e. a measure μ supported on E satisfying $\mu(B(x,r)) \leq r$ and

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \leq \mu(E).$$

Frostman's Lemma

Given a compact set $E \subset \mathbb{R}^2$ there exists a measure μ supported on E such that $\mu(E) \sim \mathcal{H}^1_\infty(E)$ and

$$\mu(B(x,r)) \leq r.$$

Step 1. Frostman measure

Let $E \subset \mathbb{R}^2$ be a compact set with PBP. Our goal is to find a non-zero measure $\mu \in \mathcal{F}(E)$, i.e. a measure μ supported on E satisfying $\mu(B(x,r)) \leq r$ and

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \leq \mu(E).$$

Frostman's Lemma

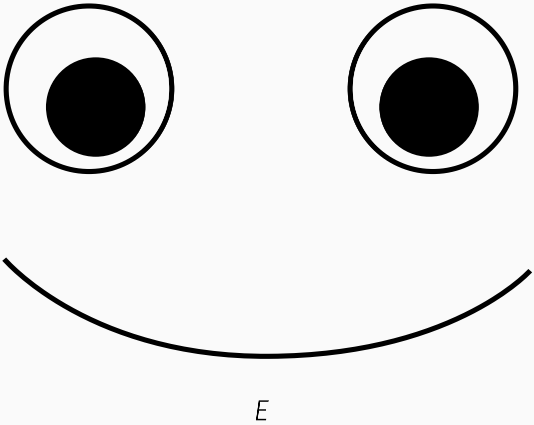
Given a compact set $E \subset \mathbb{R}^2$ there exists a measure μ supported on E such that $\mu(E) \sim \mathcal{H}^1_\infty(E)$ and

$$\mu(B(x,r)) \leq r.$$

Warning: It is **not** true that for any η with $\operatorname{supp} \eta \subset E$ and linear growth we have $\eta \in \mathcal{F}(E)$!

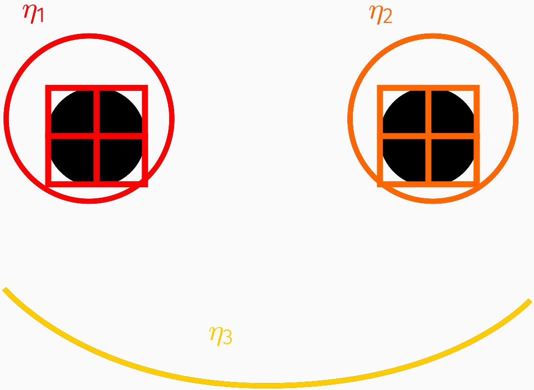
Step 2. Approximation by AD-regular measures

We perform a multi-scale approximation of the Frostman measure μ by a family of AD-regular measures $\{\eta_R\}_{R \in \mathrm{Roots}}$.



Step 2. Approximation by AD-regular measures

We perform a multi-scale approximation of the Frostman measure μ by a family of AD-regular measures $\{\eta_R\}_{R \in \mathrm{Roots}}$.



Step 2. Approximation by AD-regular measures

We perform a multi-scale approximation of the Frostman measure μ by a family of AD-regular measures $\{\eta_R\}_{R \in \mathrm{Roots}}$.

Steps 3 and 4. Approximating measures are flat

Each η_R inherits the PBP property from the set E. Since η_R are AD-regular, we get from Orponen's result

$$\iint \beta_{\eta_R,2}(x,r)^2 \frac{\eta_R(B(x,r))}{r} \frac{dr}{r} d\eta_R(x) \lesssim \eta_R(E).$$

Steps 3 and 4. Approximating measures are flat

Each η_R inherits the PBP property from the set E. Since η_R are AD-regular, we get from Orponen's result

$$\iint \beta_{\eta_R,2}(x,r)^2 \frac{\eta_R(B(x,r))}{r} \frac{dr}{r} d\eta_R(x) \lesssim \eta_R(E).$$

Summing over all $R \in$ Roots and using the fact that η_R approximate μ quite well, one can derive

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \lesssim \mu(E).$$

Hence, $\mu \in \mathcal{F}(E)$ and by the Azzam-Tolsa result

$$\gamma(E) \gtrsim \mu(E) \sim \mathcal{H}^1_\infty(E) \sim \mathrm{diam}(E).$$

■

Can we replace PBP by "uniformly large Favard length":

Question

Suppose E is compact, and for all $x \in E$ and $0 < r < \text{diam}(E)$ we have $\text{Fav}(E \cap B(x, r)) \gtrsim r$. Does this imply $\gamma(E) \gtrsim \text{diam}(E)$?

This would immediately follow from the following:

Conjecture

Suppose $E \subset [0, 1]^2$ is AD-regular. If $\text{Fav}(E) \gtrsim 1$, then there exists a Lipschitz graph Γ with $Lip(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

Questions

How does PBP relate to the "L^2-projections" property of Chang-Tolsa?

Theorem (Chang-Tolsa 2019)

Let $I \subset [0, \pi)$ be an interval. If $E \subset \mathbb{R}^2$ is compact and it supports a measure μ such that $\pi_\theta \mu \in L^2$ for a.e. $\theta \in I$, then $\gamma(E) > 0$. More precisely,

$$\gamma(E) \gtrsim \frac{\mu(E)^2}{\int_I \|\pi_\theta \mu\|_{L^2}^2 \, d\theta}.$$

Thank you!