Vitushkin's conjecture and sets with plenty of big projections

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Based on joint work with Michele Villa

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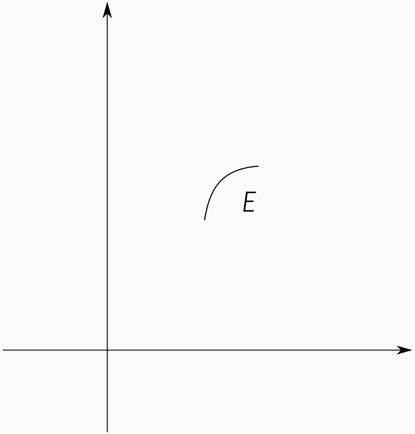
Removable sets

A compact set $E \subset \mathbb{C}$ is **removable for bounded analytic functions** if for any open $\Omega \subset \mathbb{C}$ containing E, each bounded analytic function $f : \Omega \setminus E \to \mathbb{C}$ has an analytic extension to Ω.



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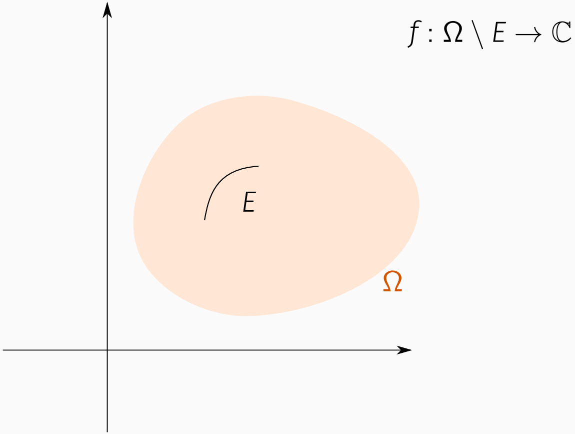

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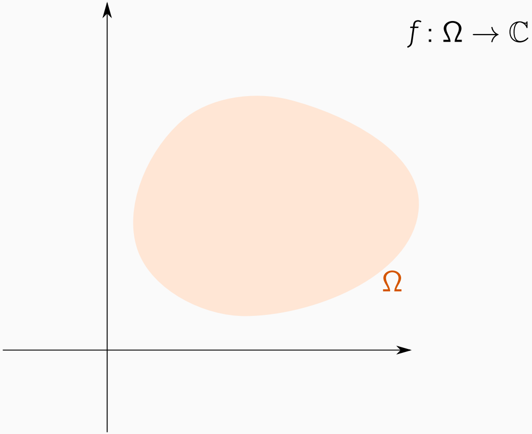

In 1947 Ahlfors characterized removability in terms of analytic capacity:

$$E \text{ is removable} \quad \Leftrightarrow \quad \gamma(E) = 0,$$

where

$$\gamma(E) = \sup\{|f'(\infty)| \; : \; f : \mathbb{C} \setminus E \to \mathbb{C} \text{ analytic}, \; \|f\|_\infty \leq 1\},$$
$$f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty)).$$

Painlevé Problem

Find a geometric characterization of removable compact sets, i.e. compact sets with $\gamma(E) = 0$.

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Question

$\gamma(E) = 0 \quad \Leftrightarrow \quad \mathcal{H}^1(E) = 0$? No!

There are sets $E \subset \mathbb{C}$ with $\gamma(E) = 0$ and $0 < \mathcal{H}^1(E) < \infty$.
(Vitushkin 1959, Garnett, Ivanov 1970s)

Vitushkin's conjecture

The sets constructed by Vitushkin, Garnett and Ivanov had very small projections. More precisely, they satisfied

$$\mathcal{H}^1(\pi_\theta(E)) = 0$$

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Define **Favard length** of E as

$$Fav(E) = \int_0^\pi \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

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$$\gamma(E) = 0 \quad \Leftrightarrow \quad Fav(E) = 0$$

Rectifiability

A set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of 1-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^1\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

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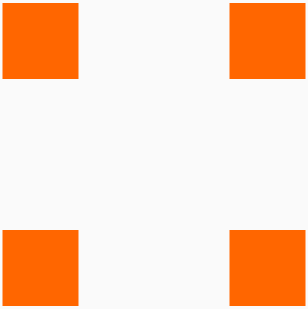
We say that $F \subset \mathbb{R}^2$ is purely unrectifiable if for every 1-dimensional Lipschitz graph Γ

$$\mathcal{H}^1(F \cap \Gamma) = 0.$$

Four-corner Cantor set



K_1

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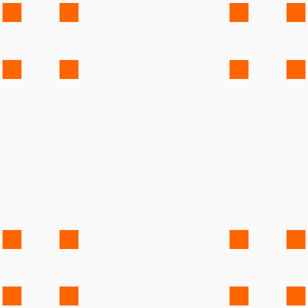

K_2

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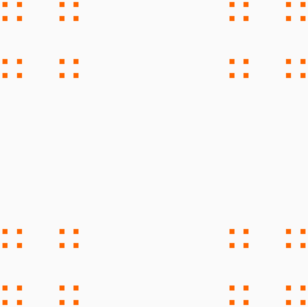

K_3

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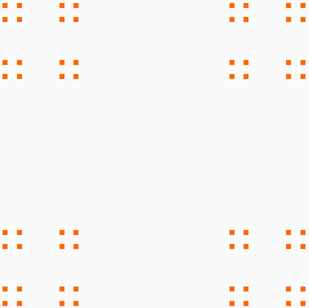

$$K = \bigcap_n K_n$$

- an elementary argument gives $0 < \mathcal{H}^1(K) < \infty$ and $Fav(K) = 0$,
- $\gamma(K) = 0$ (Garnett, Ivanov 70s)

Projections and rectifiability

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. Then

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The implication "$\Rightarrow$" is false for sets with $\mathcal{H}^1(E) = \infty$! There exist purely unrectifiable sets with $Fav > 0$ (Falconer 80s).

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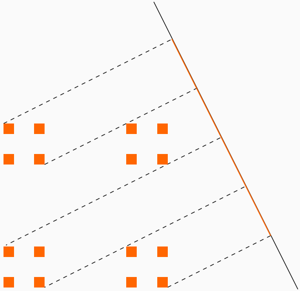

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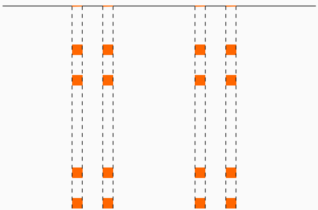

Solution to Vitushkin's conjecture

Vitushkin's conjecture

$$\gamma(E) = 0 \quad \Leftrightarrow \quad Fav(E) = 0$$

In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)

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In the case $\mathcal{H}^1(E) < \infty$ Vitushkin's conjecture is **true**! (Calderón '77, David '98)

In the case $\mathcal{H}^1(E) = \infty$, Vitushkin's conjecture is **false** (Mattila '86).

Open problems

Theorem (Jones-Murai '88)

There exists a set with $Fav(E) = 0$ and $\gamma(E) > 0$.

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Problem 1 (qualitative)

$$Fav(E) > 0 \quad \Rightarrow \quad \gamma(E) > 0?$$

Open for sets $E \subset \mathbb{C}$ with $\dim_H(E) = 1$ and non-σ-finite $\mathcal{H}^1$-measure.

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Problem 2 (quantitative)

$$\gamma(E) \gtrsim Fav(E)?$$

$$\gamma(E) \gtrsim_{Fav(E)} 1?$$

Open even in the case $\mathcal{H}^1(E) < \infty$.

What happens in the case $\mathcal{H}^1(E) < \infty$?

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Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $Fav(E) > 0$, then there exists a Lipschitz graph Γ with

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If $0 < \mathcal{H}^1(E) < \infty$ and $Fav(E) > 0$, we get

$$\gamma(E) \geq \gamma(E \cap \Gamma) \overset{\text{(Calderón '77)}}{>} 0.$$

Two big problems

If $0 < \mathcal{H}^1(E) < \infty$ and $Fav(E) > 0$, then by the Besicovitch projection theorem we get

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2. There are estimates on $\gamma(E \cap \Gamma)$ depending on $\mathcal{H}^1(E \cap \Gamma)$, e.g. if Γ is an L-Lipschitz graph, then

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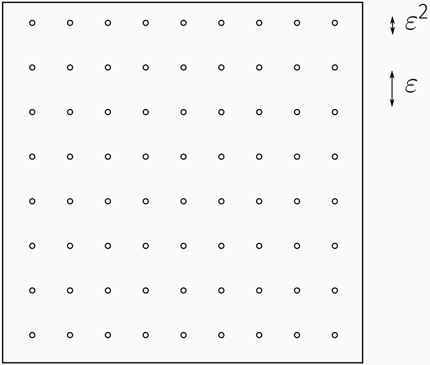
   $$\gamma(E \cap \Gamma) \gtrsim_L \mathcal{H}^1(E \cap \Gamma)...$$

   ...but the Besicovitch projection theorem gives **no quantitative bound** neither on $\mathcal{H}^1(E \cap \Gamma)$, nor on $\mathrm{Lip}(\Gamma)$!

Quantifying Besicovitch projection theorem is hard

In fact, for any $\varepsilon > 0$ there exists a set $E = E(\varepsilon) \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $Fav(E) \gtrsim 1$ such that for all L-Lipschitz graphs Γ

$$\mathcal{H}^1(E \cap \Gamma) \lesssim L\varepsilon.$$



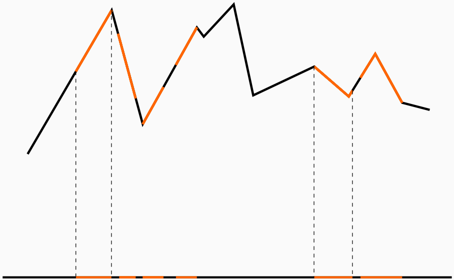
E consists of ε^{-2} uniformly distributed circles of radius ε^2.

New results

Sets with plenty of big projections

We say that a set $E \subset \mathbb{R}^2$ has **plenty of big projections** (PBP) if there exists $\delta > 0$ such that for every $x \in E$ and $0 < r < \mathrm{diam}(E)$ we have a direction $\theta_{x,r} \in [0, \pi)$ such that

$$\mathcal{H}^1(\pi_\theta(E \cap B(x,r))) \geq \delta r \quad \text{for all} \quad |\theta - \theta_{x,r}| < \delta.$$



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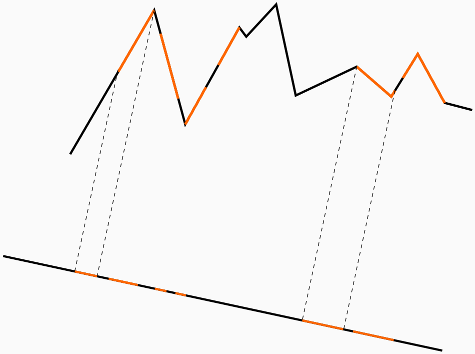

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Theorem (D.-Villa)

If a compact set $E \subset \mathbb{R}^2$ has PBP, then

$$\gamma(E) \gtrsim_\delta \mathrm{diam}(E).$$

An analogous result holds in higher dimension and codimension for capacities $\Gamma_{d,n}(E)$.

Two crucial blackboxes

Geometric characterization of γ(E)

For a set $E \subset \mathbb{R}^2$ let $\mathcal{F}(E)$ be the set of Radon measures satisfying:

- $\operatorname{supp} \mu \subset E$,

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- a flatness condition

$$\iint \beta_{\mu,2}(x, r)^2 \, \frac{\mu(B(x, r))}{r} \, \frac{dr}{r} d\mu(x) \leq \mu(E).$$

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Examples:

- $\mathcal{L}^2|_{[0,1]^2} \in \mathcal{F}([0,1]^2)$
- $\mathcal{H}^1|_\Gamma \in \mathcal{F}(\Gamma)$ for any 1-Lipschitz graph Γ
- $\mathcal{F}(K) = \{0\}$ for the 4-corner Cantor set K

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Theorem (Tolsa 2005, Azzam-Tolsa 2015)

$$\gamma(E) \sim \sup\{\mu(E) \ : \ \mu \in \mathcal{F}(E)\}$$

Takeaway: to show $\gamma(E) > 0$, it suffices to find a non-zero $\mu \in \mathcal{F}(E)$.

Big pieces of Lipschitz graphs

We say that a set $E \subset \mathbb{R}^2$ is **AD-regular** if for any $x \in E$ and $0 < r < \text{diam}(E)$ we have

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x, r)) \leq Cr.$$

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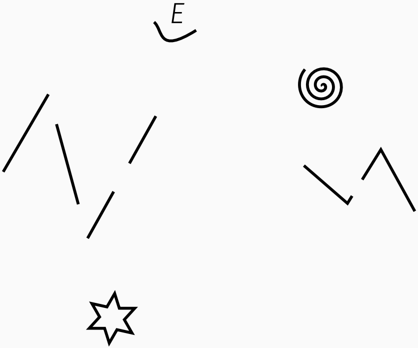
An AD-regular set E contains **big pieces of Lipschitz graphs (BPLG)** if there exist $C, L > 0$ such that for every $x \in E$ and every $0 < r < \text{diam}(E)$ there exists an L-Lipschitz graph $\Gamma = \Gamma_{x,r}$ with

$$\mathcal{H}^1(E \cap \Gamma \cap B(x,r)) \geq Cr.$$

Big pieces of Lipschitz graphs



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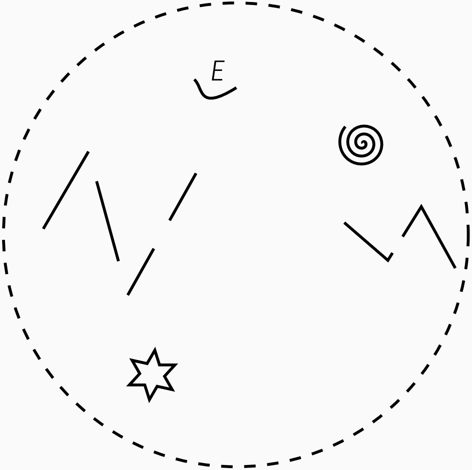

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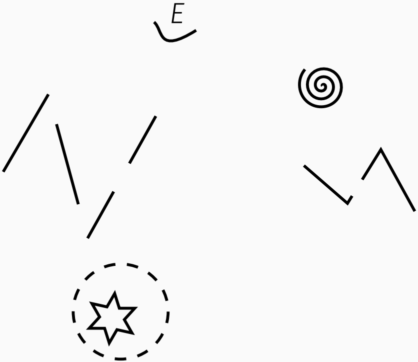

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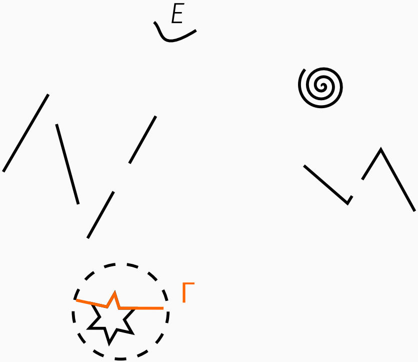

PBP and quantitative rectifiability

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Theorem (Orponen 2021)

Let E be AD-regular. If E has PBP, then it contains big pieces of Lipschitz graphs.

PBP and quantitative rectifiability

Theorem (David-Semmes 1991)

If E is AD-regular and has BPLG, then $\mu = \mathcal{H}^1|_E$ satisfies

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \lesssim \mu(E),$$

i.e. $\mu \in \mathcal{F}(E)$.

Corollary

If E is AD-regular and has PBP, then $\mu = \mathcal{H}^1|_E$ satisfies

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The proof

Theorem (D.-Villa)

If a compact set $E \subset \mathbb{R}^2$ has PBP, then

$$\gamma(E) \gtrsim_\delta \operatorname{diam}(E).$$

Plan:

1. Take the usual Frostman measure μ on E. We'll show $\mu \in \mathcal{F}(E)$.

Sketch of the proof

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3. Each η_R has PBP; apply Orponen's result to get an estimate on $\beta_{\eta_R,2}$.

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3. Each η_R has PBP; apply Orponen's result to get an estimate on $\beta_{\eta_R,2}$.
4. Transfer the estimates to $\beta_{\mu,2}$ to conclude $\mu \in \mathcal{F}(E)$.

Step 1. Frostman measure

Let $E \subset \mathbb{R}^2$ be a compact set with PBP. Our goal is to find a non-zero measure $\mu \in \mathcal{F}(E)$, i.e. a measure μ supported on E satisfying $\mu(B(x,r)) \leq r$ and

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Frostman's Lemma

Given a compact set $E \subset \mathbb{R}^2$ there exists a measure μ supported on E such that $\mu(E) \sim \mathcal{H}^1_\infty(E)$ and

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Given a compact set $E \subset \mathbb{R}^2$ there exists a measure μ supported on E such that $\mu(E) \sim \mathcal{H}^1_\infty(E)$ and

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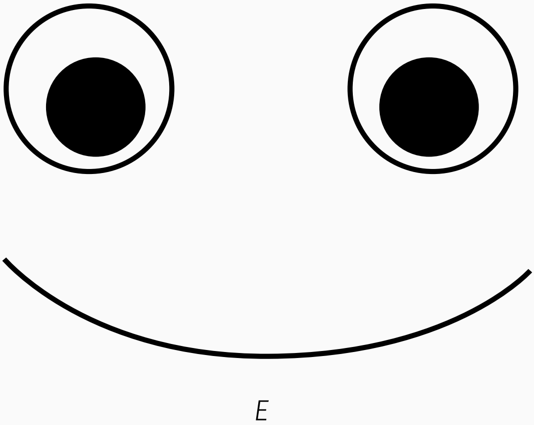
Warning: It is **not** true that for any η with $\operatorname{supp} \eta \subset E$ and linear growth we have $\eta \in \mathcal{F}(E)$!

Step 2. Approximation by AD-regular measures

We perform a multi-scale approximation of the Frostman measure μ by a family of AD-regular measures $\{\eta_R\}_{R \in \mathrm{Roots}}$.



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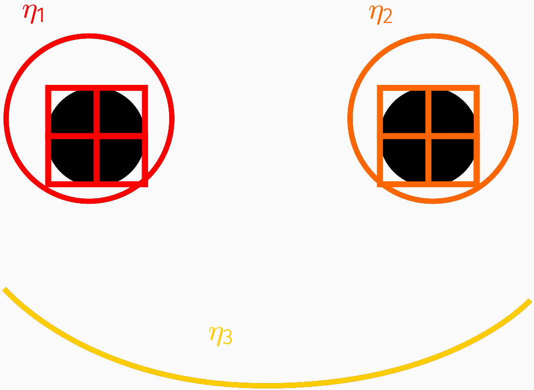

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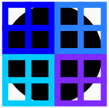

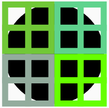

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In technical terms:

- We decompose the David-Christ lattice $\mathcal{D}$ into a family of trees $\{\mathcal{T}(R)\}_{R\in\mathrm{Roots}}$ such that μ "is AD-regular in each $\mathcal{T}(R)$."
- To do that, we conduct a stopping time argument involving a single condition:

$$Q \in \mathrm{LD}(R) \quad \Leftrightarrow \quad \frac{\mu(Q)}{\ell(Q)} \le 0.1\,\frac{\mu(R)}{\ell(R)}.$$

- We construct an approximating measure η_R for each $\mathcal{T}(R)$.

Steps 3 and 4. Approximating measures are flat

Each η_R inherits the PBP property from the set E. Since η_R are AD-regular, we get from Orponen's result

$$\iint \beta_{\eta_R,2}(x,r)^2 \frac{\eta_R(B(x,r))}{r} \frac{dr}{r} d\eta_R(x) \lesssim \eta_R(E).$$

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Summing over all $R \in$ Roots and using the fact that η_R approximate μ quite well, one can derive

$$\iint \beta_{\mu,2}(x,r)^2 \frac{\mu(B(x,r))}{r} \frac{dr}{r} d\mu(x) \lesssim \sum_{R \in \text{Roots}} \left(\frac{\mu(R)}{\ell(R)} \right)^2 \mu(R)$$

$$\lesssim \mu(E).$$

Hence, $\mu \in \mathcal{F}(E)$ and by the Azzam-Tolsa result

$$\gamma(E) \gtrsim \mu(E) \sim \mathcal{H}^1_\infty(E) \sim \text{diam}(E).$$

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Questions

Can we replace PBP by "uniformly large Favard length": for all $x \in E$ and $0 < r < \text{diam}(E)$ we have $Fav(E \cap B(x, r)) \gtrsim r$. This would immediately follow from the following:

Conjecture

Suppose $E \subset [0, 1]^2$ is AD-regular. If $Fav(E) \gtrsim 1$, then there exists a Lipschitz graph Γ with $Lip(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

How does PBP relate to the "L^2-projections" property of Chang-Tolsa?

Theorem (Chang-Tolsa 2019)

Let $I \subset [0, \pi)$ be an interval. If $E \subset \mathbb{R}^2$ is compact and it supports a measure μ such that $\pi_\theta \mu \in L^2$ for a.e. $\theta \in I$, then $\gamma(E) > 0$. More precisely,

$$\gamma(E) \gtrsim \frac{\mu(E)^2}{\int_I \|\pi_\theta \mu\|_{L^2}^2 \, d\theta}.$$

Thank you!