Quantifying Besicovitch projection theorem

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Rectifiability

A set $E \subset \mathbb{R}^2$ is **rectifiable** if there exists a countable number of 1-dimensional Lipschitz graphs Γ_i such that

$$\mathcal{H}^1\left(E \setminus \bigcup_i \Gamma_i\right) = 0.$$

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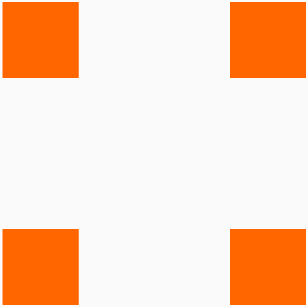
We say that $F \subset \mathbb{R}^2$ is purely unrectifiable if for every 1-dimensional Lipschitz graph Γ

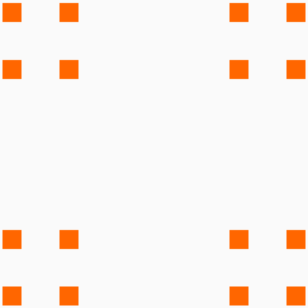
$$\mathcal{H}^1(F \cap \Gamma) = 0.$$

Four-corner Cantor set



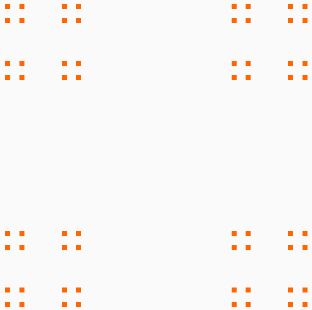
K_1

Four-corner Cantor set



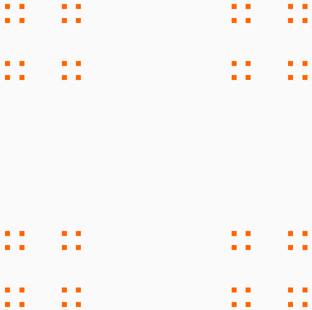
K_2

Four-corner Cantor set



K_3

Four-corner Cantor set



$$K = \bigcap_n K_n$$

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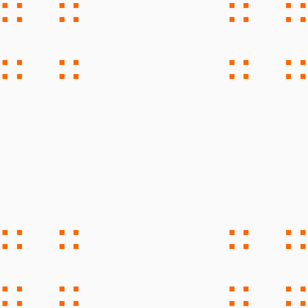


$$K = \bigcap_n K_n$$

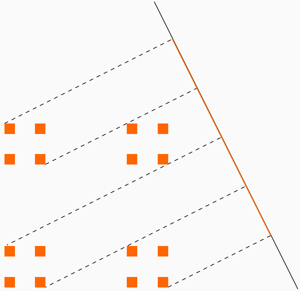
Fact

Any set E with $0 < \mathcal{H}^1(E) < \infty$ can be decomposed $E = R \cup U$ with R rectifiable and U purely unrectifiable.

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. Then, E is purely unrectifiable if and only if

$$\mathcal{H}^1(\pi_\theta(E)) = 0 \quad \text{for a.e. } \theta \in (0, \pi).$$



Projections and rectifiability

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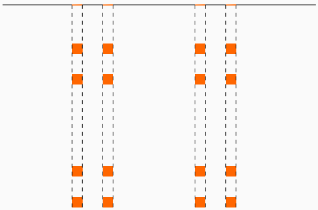

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We are interested in quantifying this result.

Previously studied by Mattila, David, Semmes, Tao, Łaba, Zhai, Bateman, Volberg, Bond, Zahl, Nazarov, Wilson, Martikainen, Orponen, Bongers, Taylor, Marshall, Zhang, Vardakis.

Define Favard length of E as

$$\mathrm{Fav}(E) = \int_0^{\pi} \mathcal{H}^1(\pi_\theta(E))\, d\theta.$$

Theorem (Besicovitch 1939)

Let $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$. If $\mathrm{Fav}(E) > 0$, then there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

Quantifying Besicovitch's theorem

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Problem

Can we quantify the dependence of $\mathrm{Lip}(\Gamma)$ and $\mathcal{H}^1(E \cap \Gamma)$ on $\mathrm{Fav}(E)$?

Why is this interesting?

- fits into the framework of the quantitative rectifiability field, connections to PDEs and SIOs
- seems necessary for the solution of Vitushkin's conjecture

Naive conjecture...

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Naive conjecture

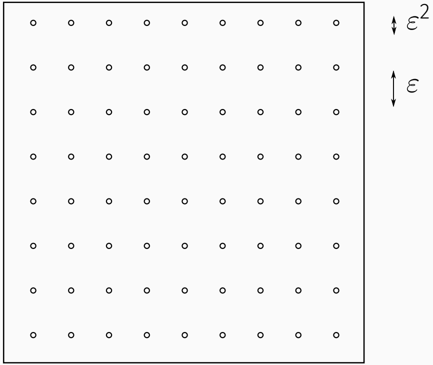
Let $E \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$. Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

… is false

For any $\varepsilon > 0$ there exists a set $E = E_\varepsilon \subset [0,1]^2$ with $\mathcal{H}^1(E) \sim 1$ and $\mathsf{Fav}(E) \gtrsim 1$ such that for all L-Lipschitz graphs Γ

$$\mathcal{H}^1(E \cap \Gamma) \lesssim L\varepsilon.$$



E consists of ε^{-2} uniformly distributed circles of radius ε^2.

Reasonable conjecture

We say that a set $E \subset \mathbb{R}^2$ is **Ahlfors regular** if for any $x \in E$ and $0 < r < \mathrm{diam}(E)$ we have

$$C^{-1} r \leq \mathcal{H}^1(E \cap B(x,r)) \leq Cr.$$

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Conjecture

Let $E \subset [0,1]^2$ be an Ahlfors regular set with $\mathcal{H}^1(E) \sim 1$ and $\text{Fav}(E) \gtrsim 1$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\text{Lip}(\Gamma) \lesssim 1$ and

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Previous results

Conjecture

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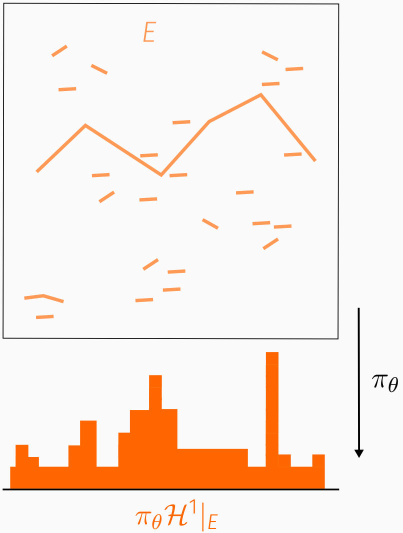
Significant progress towards the conjecture due to:

- Orponen 2021: sets with "plenty of big projections,"
- Martikainen-Orponen 2018: sets with projections in L^2.

Sets with projections in L^2

Big projections vs projections in L^p

Denote by $\pi_\theta \mathcal{H}^1|_E$ the pushforward of $\mathcal{H}^1|_E$ by π_θ.



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Observation

If $\pi_\theta \mathcal{H}^1|_E \in L^p$ for some $p > 1$, then

$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim \frac{\mathcal{H}^1(E)^{p'}}{\|\pi_\theta \mathcal{H}^1|_E\|_p^{p'}}.$$

Indeed:

$$\mathcal{H}^1(E) = \int_{\pi_\theta(E)} \pi_\theta \mathcal{H}^1|_E(x)\,dx$$

$$\leq \left(\int \pi_\theta \mathcal{H}^1|_E(x)^p\,dx \right)^{1/p} \mathcal{H}^1(\pi_\theta(E))^{1/p'}.$$

Sets with projections in L^2

Theorem (Martikainen-Orponen 2018)

Let $E \subset [0,1]^2$ be an Ahlfors regular set with $\mathcal{H}^1(E) \sim 1$. Suppose that there exists an arc $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that

$$\int_G \|\pi_\theta \mathcal{H}^1|_E\|_{L^2}^2 \, d\theta \lesssim 1.$$

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

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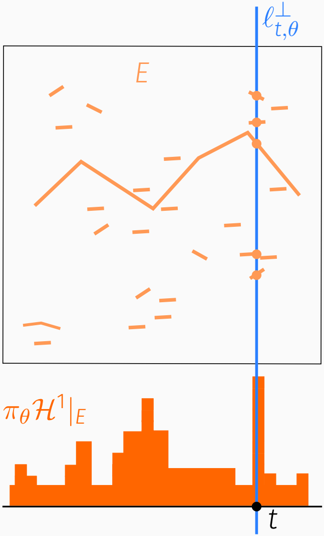
Projections in L^2 are special



$$\int_{\pi_\theta(E)} |\pi_\theta \mathcal{H}^1|_E(t)|^2 \, dt \sim \int_{\pi_\theta(E)} \#\{\ell_{t,\theta}^{\perp} \cap E\}^2 \, dt$$

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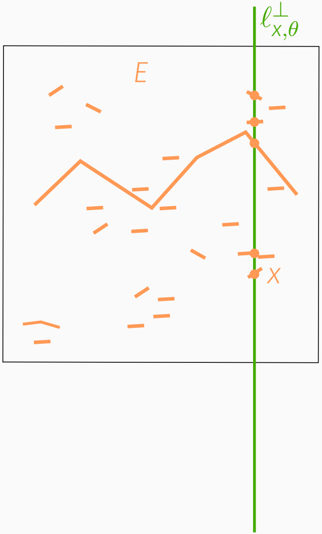

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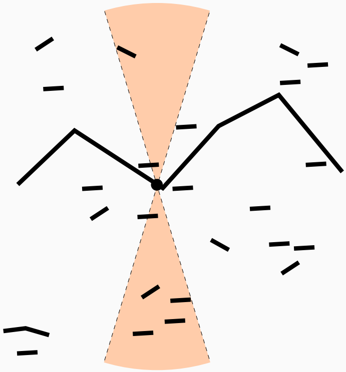

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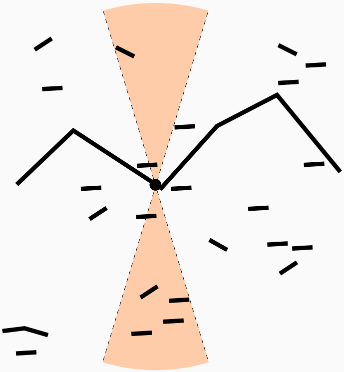
$$\sim \int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x,G,r))}{r} \frac{dr}{r} \, dx.$$



$$X(x,G) = \bigcup_{\theta \in G} \ell_{x,\theta}^\perp$$

$$X(x,G,r) = X(x,G) \cap B(x,r)$$

Projections in L^2 are special

In fact, one can use Fourier analysis to show:

Theorem (Chang-Tolsa 2020)

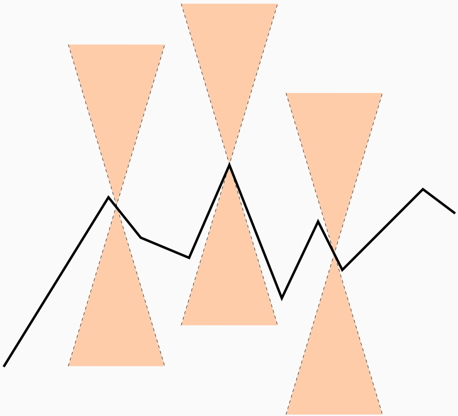
Let μ be a finite, compactly supported measure on $\mathbb{R}^2$, and $G \subset \mathbb{S}^1$ an open set. Then,

$$\int\int_0^\infty \frac{\mu(X(x,G,r))}{r}\frac{dr}{r}\,d\mu(x) \lesssim \int_G \|\pi_\theta \mu\|_{L^2}^2\,d\theta.$$

Why is this useful?

Recall: $E \subset \mathbb{R}^2$ is a subset of a Lipschitz graph iff there exists an open cone X such that

$$x \in E \quad \Rightarrow \quad E \cap X(x) = \varnothing.$$



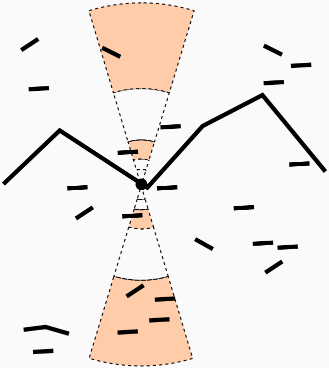
Why is this useful?

In our setting, the estimate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G, r))}{r} \frac{dr}{r} \, dx \lesssim 1$$

can be used to show that for most $x \in E$

$$\#\{j \in \mathbb{Z} \;:\; E \cap X(x, G, 2^{-j-1}, 2^{-j}) \neq \varnothing\} \lesssim 1.$$



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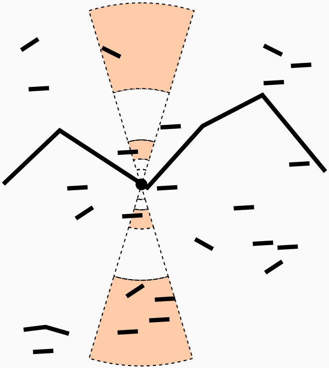

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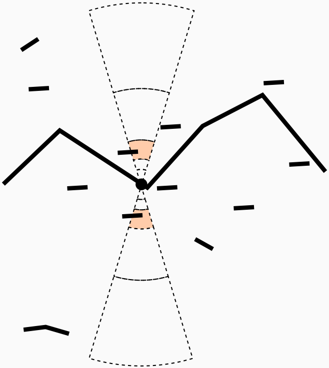

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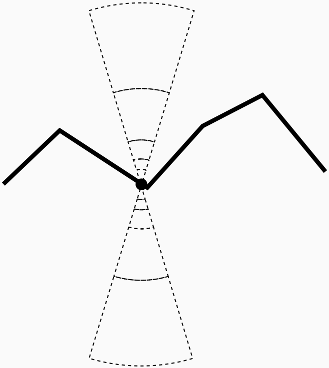

New result

Theorem (D. 2022)

Let $E \subset [0,1]^2$ be an Ahlfors regular set with $\mathcal{H}^1(E) \sim 1$. Suppose that there exists a measurable $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that

$$\|\pi_\theta \mathcal{H}^1|_E\|_{L^\infty} \lesssim 1 \quad \text{for } \theta \in G.$$

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

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Why is this significant?

Conjecture

Let $E \subset [0,1]^2$ be an Ahlfors regular set with $\mathcal{H}^1(E) \sim 1$ and $\mathrm{Fav}(E) \gtrsim 1$.

Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1.$$

Note that $\mathrm{Fav}(E) \gtrsim 1$ if and only if there exists a measurable $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ such that

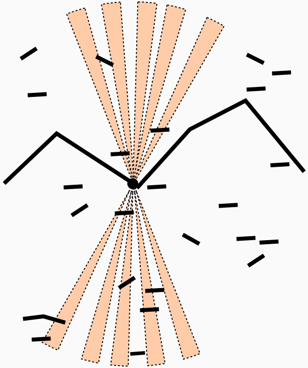
$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim 1 \quad \text{for } \theta \in G.$$

Difficulty 1. We can still get the estimate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G, r))}{r} \frac{dr}{r} \, dx \lesssim 1,$$

but now we cannot transform it to

$$\{j \in \mathbb{Z} \ : \ E \cap X(x, G, 2^{-j-1}, 2^{-j}) \neq \varnothing\} \lesssim 1.$$



New difficulties

Recall: $E \subset \mathbb{R}^2$ is a subset of a Lipschitz graph iff there exists an open cone X such that

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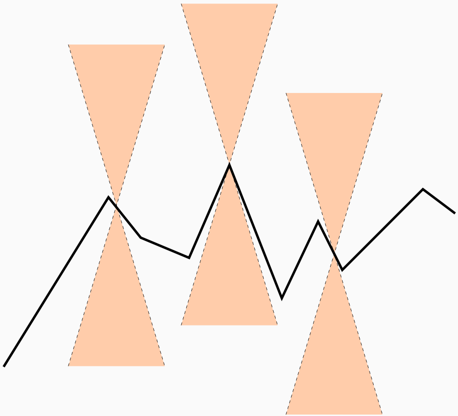
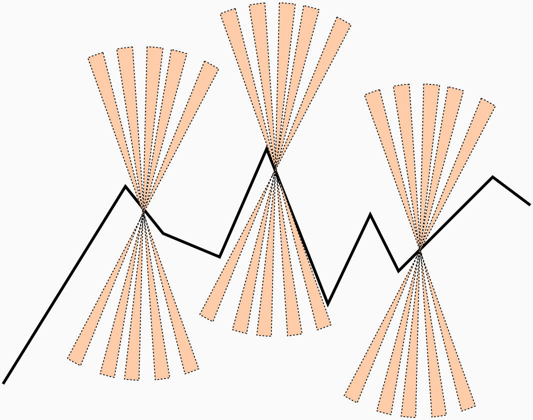


Difficulty 2. We are missing a characterization of Lipschitz graphs in terms of the “irregular, star-shaped” cones.



New difficulties

Difficulty 2. We are missing a characterization of Lipschitz graphs in terms of the "irregular, star-shaped" cones.

Question

Suppose that $E \subset [0,1]^2$ is Ahlfors regular with $\mathcal{H}^1(E) \sim 1$, and satisfies

$$x \in E \quad \Rightarrow \quad E \cap X(x, G) = \varnothing$$

for some $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$.

- Is E rectifiable?
- Is there a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1?$$

About the proof

Idea of the proof

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Let $E \subset [0,1]^2$ be an Ahlfors regular set with $\mathcal{H}^1(E) \sim 1$. Suppose that there exists a measurable $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that

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Then, there exists a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and

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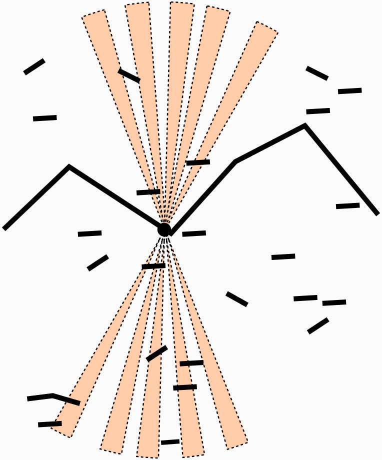
We prove that there exists an arc $J \subset \mathbb{S}^1$ with $\mathcal{H}^1(J) \sim 1$ such that

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, J, r))}{r} \frac{dr}{r} \, dx \lesssim 1.$$

Then, we can use the result of Martikainen-Orponen to find the desired big piece of a Lipschitz graph.

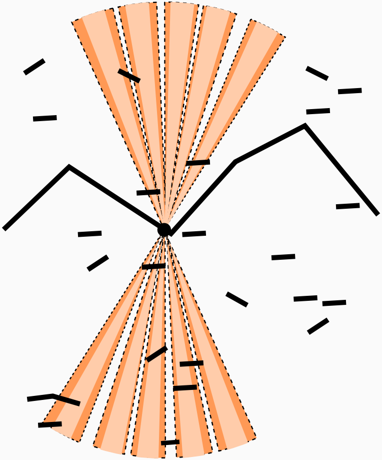
Good directions propagate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G, r))}{r} \frac{dr}{r} \, dx \lesssim 1.$$



Good directions propagate

$$\int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G', r))}{r} \frac{dr}{r}\, dx \lesssim \int_E \int_0^\infty \frac{\mathcal{H}^1(E \cap X(x, G, r))}{r} \frac{dr}{r}\, dx \lesssim 1.$$



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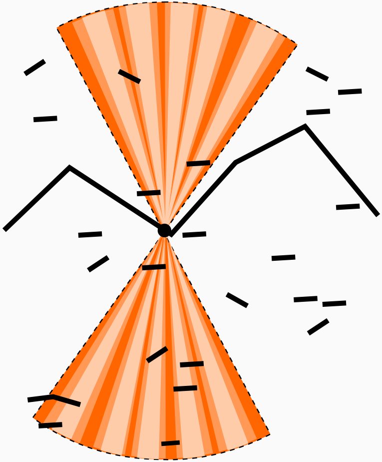

Good directions propagate

Main propositon

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- $J \subset \mathbb{S}^1$ is an arc, and $G_J \subset J$ is measurable with $\mathcal{H}^1(J \setminus G_J) \leq \varepsilon \mathcal{H}^1(J)$,

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- technical assumptions involving $\|\pi_\theta \mathcal{H}^1|_E\|_\infty$.

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- technical assumptions involving $\|\pi_\theta \mathcal{H}^1|_E\|_\infty$.

Then,

$$\int_E \int_0^1 \frac{\mathcal{H}^1(E \cap X(x, 3J, r))}{r} \frac{dr}{r} d\mathcal{H}^1(x) \lesssim \int_E \int_0^1 \frac{\mathcal{H}^1(E \cap X(x, G_J, r))}{r} \frac{dr}{r} d\mathcal{H}^1(x) + \mathcal{H}^1(J).$$

Questions

Questions

Can we relax the L^∞-assumption to the L^2-assumptions?

Question 1

Let $E \subset [0,1]^2$ be an Ahlfors regular set with $\mathcal{H}^1(E) \sim 1$. Suppose that there exists a measurable $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that

$$\|\pi_\theta \mathcal{H}^1|_E\|_{L^2} \lesssim 1 \quad \text{for } \theta \in G.$$

Does there exist a Lipschitz graph $\Gamma \subset \mathbb{R}^2$ with $\mathbf{Lip}(\Gamma) \lesssim 1$ and

$$\mathcal{H}^1(E \cap \Gamma) \gtrsim 1?$$

Questions

Can a similar approach be used to prove the full conjecture?

Question 2

Suppose that $E \subset [0,1]^2$ is an Ahflors regular set with $\mathcal{H}^1(E) = 1$. Does there exist $\varepsilon > 0$ and $\delta_0 > 0$ such that, if

- $J \subset \mathbb{S}^1$ is an arc with $\mathcal{H}^1(J) \leq \delta_0$, and $G_J \subset J$ is measurable with $\mathcal{H}^1(J \setminus G_J) \leq \varepsilon \mathcal{H}^1(J)$,
- for some $\delta_0^{-1}\mathcal{H}^1(J) \leq C \leq 1$

$$\mathcal{H}^1(\pi_\theta(E)) \geq C \quad \text{for } \theta \in G_J,$$

then

$$\mathcal{H}^1(\pi_\theta(E)) \gtrsim_C 1 \quad \text{for } \theta \in 3J?$$

Thank you!